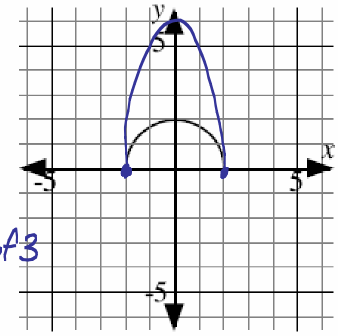
Transformations Lesson #7:
Expansions and Compressions about the x- or y- axis Part 1

Warm-Up #1 *Comparing the Graphs of $y = f(x)$ and $y = af(x)$, where $a > 0$*

The graph of $y = f(x) = \sqrt{4 - x^2}$ is shown.

a) Write an equation which represents $y = 3f(x)$.

$y = 3\sqrt{4 - x^2}$



b) Use a graphing calculator to sketch $y = 3f(x)$ on the grid.

c) Describe how the number 3 in $y = 3f(x)$ affects:

- the general sketch of $y = f(x)$
  (i.e. whether it expands or compresses $y = f(x)$)

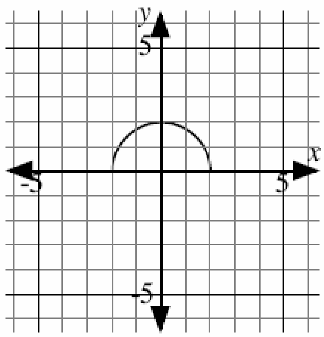
  expands vertically about the x axis by a factor of 3
- the x-intercepts of the graph of $y = f(x)$

  no change
- the y-intercept of the graph of $y = f(x)$.

  multiplied by 3

d) Write an equation which represents $y = \frac{1}{2}f(x)$.

$y = \frac{1}{2}\sqrt{4 - x^2}$



e) Use a graphing calculator to sketch $y = \frac{1}{2}f(x)$ on the grid.

f) Describe how the number $\frac{1}{2}$ in $y = \frac{1}{2}f(x)$ affects:

- the general sketch of $y = f(x)$
  (i.e. whether it expands or compresses $y = f(x)$)

  compresses vertically about the x axis by a factor of $\frac{1}{2}$
- the x-intercepts of the graph of $y = f(x)$

  no change
- the y-intercept of the graph of $y = f(x)$.

  multiplied by ½

g) Compared to the graph of $y = f(x)$, the graph of $y = af(x)$ results in a <u>Vertical</u> stretch about the <u>x</u>-axis by a factor of a.

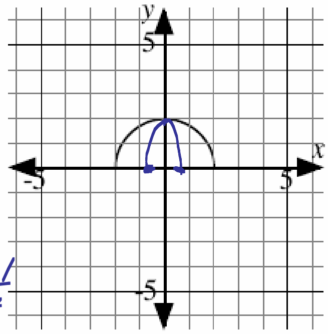
- If $a > 1$, the stretch is an <u>expansion</u> .
- If $0 < a < 1$, the stretch is a <u>compression</u> .

Warm-Up #2 ***Comparing the Graphs of $y = f(x)$ and $y = f(bx)$, where $b > 0$***

The graph of $y = f(x) = \sqrt{4 - x^2}$ is shown.

a) Write an equation which represents $y = f(4x)$.

$y = \sqrt{4-(4x)^2}$ $\quad y = \sqrt{4-16x^2}$



b) Use a graphing calculator to sketch $y = f(4x)$ on the grid.

c) Describe how the number 4 in $y = f(4x)$ affects:

- the general sketch of $y = f(x)$
(i.e. whether it expands or compresses $y = f(x)$)

compresses horizontally about the y axis by a factor of $\frac{1}{4}$

- the x-intercepts of the graph of $y = f(x)$

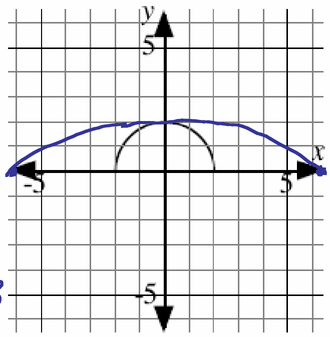
multiplied by $\frac{1}{4}$

- the y-intercept of the graph of $y = f(x)$.

no change

d) Write an equation which represents $y = f\left(\frac{1}{3}x\right)$.

$y = \sqrt{4-\left(\frac{1}{3}x\right)^2}$ $\quad y = \sqrt{4-\frac{1}{9}x^2}$



e) Use a graphing calculator to sketch $y = f\left(\frac{1}{3}x\right)$ on the grid.

f) Describe how the number $\frac{1}{3}$ in $y = f\left(\frac{1}{3}x\right)$ affects:

- the general sketch of $y = f(x)$
(i.e. whether it expands or compresses $y = f(x)$)

expands horizontally about the y-axis by a factor of 3

- the x-intercepts of the graph of $y = f(x)$

multiplied by 3

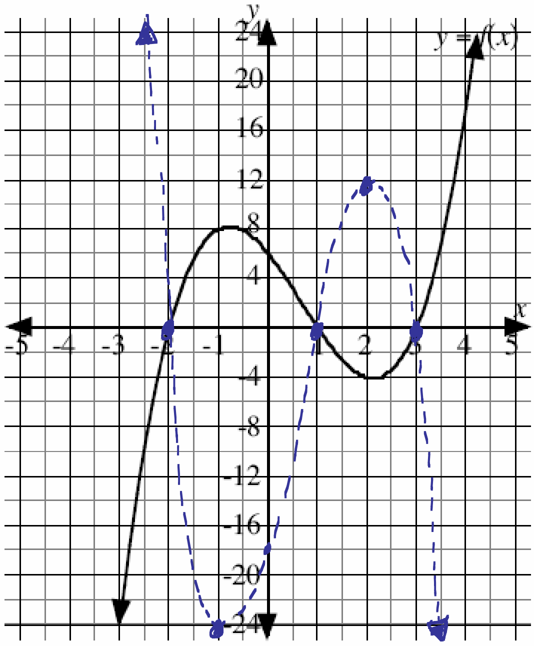
- the y-intercept of the graph of $y = f(x)$. no change

g) Compared to the graph of $y = f(x)$, the graph of $y = f(bx)$
results in a horizontal stretch about the y-axis by a factor of $\frac{1}{b}$.

- If $b > 1$, the stretch is a compression.
- If $0 < b < 1$, the stretch is an expansion.

Warm-Up #3 ***Comparing the Graphs of $y = f(x)$ and $y = af(x)$, where $a < 0$***

The graph of $y = f(x) = x^3 - 2x^2 - 5x + 6$ is shown.



a) Write an equation which represents $y = -3f(x)$.

$y = -3(x^3 - 2x^2 - 5x + 6)$
$= -3x^3 + 6x^2 + 15x - 18$

b) Use a graphing calculator to sketch $y = -3f(x)$.

c) Describe how the number -3 in $y = -3f(x)$ affects:

- the general sketch of $y = f(x)$

  - expands vertically by a factor of 3
  - reflected in the x-axis

- the x-intercepts of the graph of $y = f(x)$

  no change

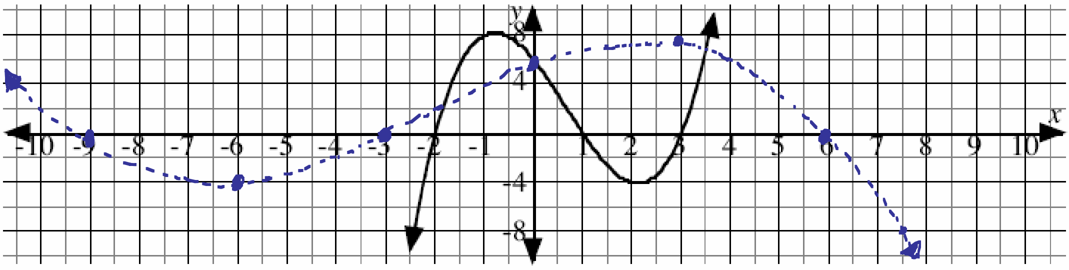
- the y-intercept of the graph of $y = f(x)$.

  multiplied by -3

d) Compared to the graph of $y = f(x)$, the graph of $y = af(x)$, where $a < 0$, results in a vertical stretch about the x-axis by a factor of $|a|$ together with a reflection in the x -axis.

Warm-Up #4 ***Comparing the Graphs of $y = f(x)$ and $y = f(bx)$, where $b < 0$***

The graph of $y = f(x) = x^3 - 2x^2 - 5x + 6$ is shown.



a) Write an equation which represents $y = f\left(-\frac{1}{3}x\right)$.

$$y = \left(-\frac{1}{3}x\right)^3 - 2\left(-\frac{1}{3}x\right)^2 - 5\left(-\frac{1}{3}x\right) + 6$$

$$= -\frac{1}{27}x^3 - \frac{2}{9}x^2 + \frac{5}{3}x + 6$$

b) Use a graphing calculator to sketch $y = f\left(-\frac{1}{3}x\right)$.

c) Describe how the number $-\frac{1}{3}$ in $y = f\left(-\frac{1}{3}x\right)$ affects:

- the general sketch of $y = f(x)$

- expanded horizontally by a factor of 3
- reflected in the y-axis

- the x-intercepts of the graph of $y = f(x)$

multiplied by 3

- the y-intercept of the graph of $y = f(x)$.

no change

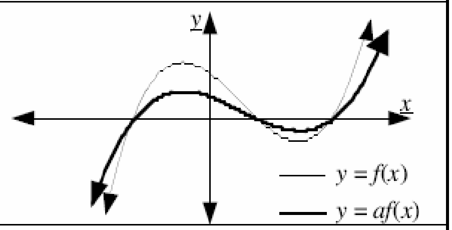
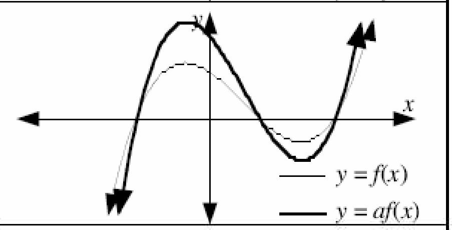
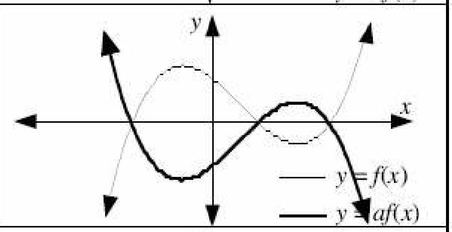
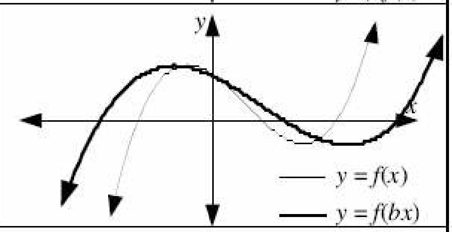
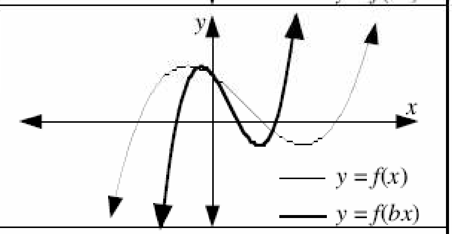
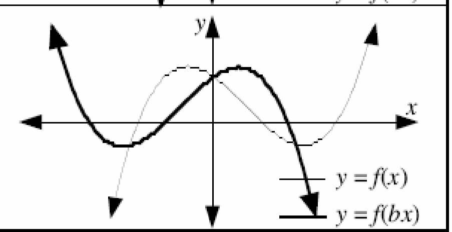
d) Compared to the graph of $y = f(x)$, the graph of $y = f(bx)$, where $b < 0$,

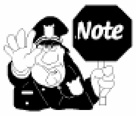
results in a <u>horizontal</u> stretch about the <u>y</u>-axis by a factor of $\frac{1}{|b|}$ together with

a reflection in the <u>y</u> -axis.

Expansions and Compressions

An expansion or a compression on a graph are transformations which stretch the graph vertically or horizontally.
The graph of $y = f(x)$ and the graph of $y = af(x)$ or $y = f(bx)$ is given.
Fill in the blanks in the table.

a or b	*Expansion or Compression*	*Graph*
$0 < a < 1$	The graph of $y = f(x)$ will be compressed vertically by a factor of a about the x-axis.	
$a > 1$	The graph of $y = f(x)$ will be expanded vertically by a factor of a about the x-axis. .	
$a < 0$	The graph of $y = f(x)$ will be reflected in the x-axis and expanded or compressed vertically about the x-axis.	
$0 < b < 1$	The graph of $y = f(x)$ will be expanded horizontally by a factor of $\frac{1}{b}$ about the y-axis.	
$b > 1$	The graph of $y = f(x)$ will be compressed horizontally by a factor of $\frac{1}{b}$ about the y-axis..	
$b < 0$	The graph of $y = f(x)$ will be reflected in the y axis and expanded or compressed horizontally about the y-axis.	



$y = af(x)$ can be written as $\frac{1}{a}y = f(x)$.

Given the function $y = f(x)$:

- **replacing x with bx, (i.e. $x \to bx$) describes a horizontal stretch about the y-axis.**
  i.e. $y = f(bx)$ describes a horizontal stretch.

- **replacing y with $\frac{1}{a}y$, (i.e. $y \to \frac{1}{a}y$) describes a vertical stretch about the x-axis.**
  i.e. $\frac{1}{a}y = f(x)$ or $y = af(x)$ describes a vertical stretch.

In general, if $\frac{1}{a}y = f(bx)$ or $y = af(bx)$, then for

$a > 1$ there is a vertical expansion
$0 < a < 1$ there is a vertical compression
$a < 0$ there is also a reflection in the x-axis
$b > 1$ there is a horizontal compression
$0 < b < 1$ there is a horizontal expansion
$b < 0$ there is also a reflection in the y-axis

Class Ex. #1

Write the replacement for x or y and write the equation of the image of $y = f(x)$ after each transformation.

$y = af(bx)$

a) A horizontal expansion by a factor of 6 about the y-axis. $b = \frac{1}{6}$

$y = f(\frac{1}{6}x)$

b) A vertical compression by a factor of $\frac{1}{5}$ about the x-axis. $a = \frac{1}{5}$

$y = \frac{1}{5}f(x)$

c) A reflection in the x-axis and a vertical expansion about the x-axis by a factor of 3. $a = -3$

$y = -3f(x)$

d) A horizontal compression about the y-axis by a factor of $\frac{1}{2}$ and $b = 2$
a vertical compression about the x-axis by a factor of $\frac{1}{4}$. $a = \frac{1}{4}$

$y = \frac{1}{4}f(2x)$

Class Ex. #2

How does the graph of $3y = f(x)$ compare with the graph of $y = f(x)$?

$\frac{3y}{3} = \frac{f(x)}{3}$

$y = \frac{1}{3}f(x)$

$a = \frac{1}{3} \Rightarrow$ Vertical compression by a factor of $\frac{1}{3}$ about the x-axis

Class Ex. #3

What happens to the graph of the function $y = f(x)$ if you make these changes?

a) Replace x with $4x$. hor comp by factor of $\frac{1}{4}$

$b = 4$ factor $\frac{1}{4}$

b) Replace y with $\frac{1}{3}y$. Vert expand by a factor of 3

$a = 3$ factor 3

c) Replace y with $6y$ and x with $\frac{1}{3}x$. vert comp by a factor of $\frac{1}{6}$

hor expand by a factor of 3

$a = \frac{1}{6}$ factor $\frac{1}{6}$

$b = \frac{1}{3}$ factor 3

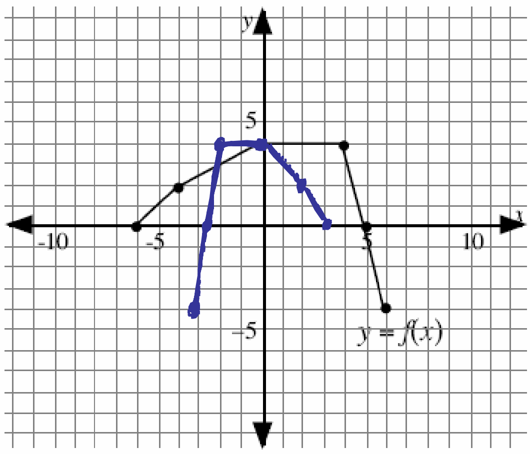
Class Ex. #4

The graph of $y = f(x)$ is shown.

Sketch $y = f(-2x)$.

$b = -2$ factor $-\frac{1}{2}$

- hor comp by a factor of ½
- reflection in the y axis



Complete Assignment Questions #1 - #7

Assignment

1. Write the replacement for x or y and write the equation of the image of $y = f(x)$ after each transformation.

a) A horizontal expansion by a factor of 3 about the y-axis.

b) A vertical expansion by a factor of 6 about the x-axis.

c) A horizontal compression about the y-axis by a factor of $\frac{5}{7}$.

d) A vertical compression about the x-axis by a factor of $\frac{2}{3}$.

e) A reflection in the y-axis and a horizontal expansion by a factor of 3 about the y-axis.

f) A reflection in the x-axis and a vertical compression by a factor of $\frac{3}{4}$ about the x-axis.

g) A reflection in the y-axis and a horizontal compression about the y-axis by a factor of $\frac{3}{4}$.

h) A horizontal expansion about the y-axis by a factor of 4 and a vertical expansion about the x-axis by a factor of 4.

i) A horizontal compression about the y-axis by a factor of 0.5, a vertical expansion by a factor of 2 about the x-axis and a reflection in the x-axis.

2. The function $y = f(x)$ is transformed to $y = af(bx)$. Determine the values of a and b for:

a) A horizontal compression by a factor of $\frac{2}{3}$ about the y-axis.

b) A vertical expansion about the x-axis by a factor of 5.

c) A horizontal expansion about the y-axis by a factor of $\frac{5}{2}$ and a reflection in the y-axis.

d) A vertical compression about the x-axis by a factor of $\frac{1}{3}$, a horizontal compression about the y-axis by a factor of $\frac{1}{10}$ and a reflection in the y-axis.

3. Consider the function $f(x) = x^2$.

a) Determine the equation of the image of the function if it is expanded vertically by a factor of 4 about the x-axis.

b) Determine the equation of the image of the function if it is compressed horizontally by a factor of $\frac{1}{2}$ about the y-axis.

c) What do you notice?

d) Give an example of a function where the stretches in a) and b) would not result in the same image.

4. **a)** What information about the graph of $y = f(kx)$ does k provide?

b) What information about the graph of $ky = f(x)$ does k provide?

c) What information about the graph of $y - k = f(x)$ does k provide?

d) What information about the graph of $y = f(x - k)$ does k provide?

e) What information about the graph of $y = kf(x)$ does k provide?

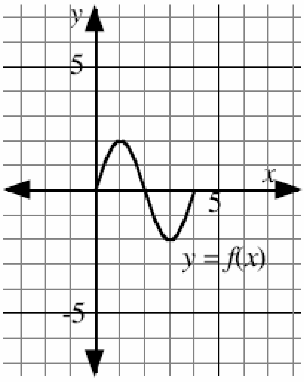
5. The graph of $y = f(x)$ is shown. In each case:

i) sketch the graph of the transformed function

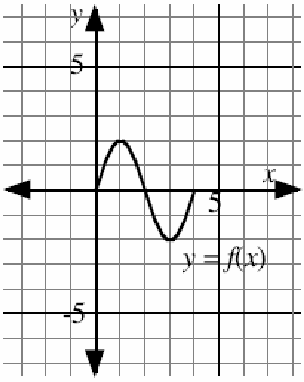
ii) state the domain and range of the transformed function

iii) state the coordinates of any invariant points.

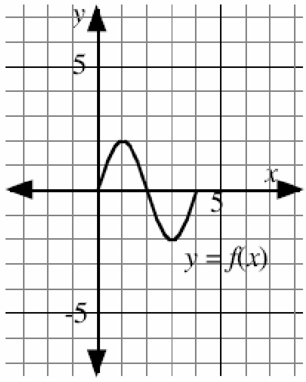
a) $y = f(2x)$



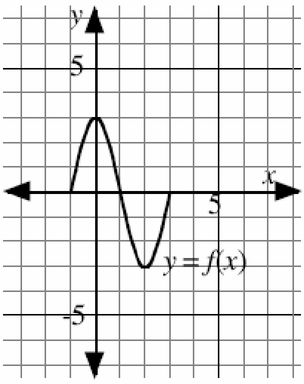
b) $y = -2f(x)$



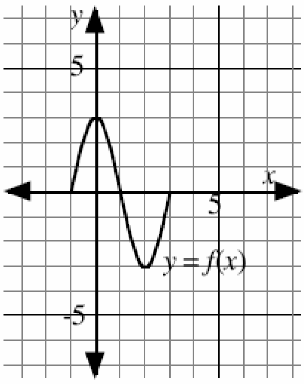
c) $y = \frac{1}{2}f\left(\frac{1}{2}x\right)$



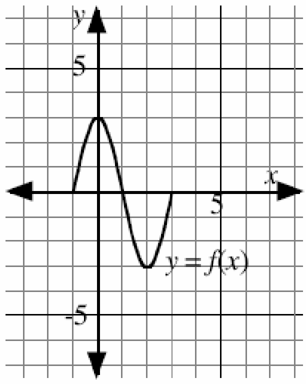
d) $y = f(2x)$



e) $y = -2f(x)$



f) $y = \frac{1}{2}f\left(\frac{1}{2}x\right)$



6. What happens to the graph of the function $y = f(x)$ if the following replacements are made?

a) Replace x with $\frac{1}{2}x$. **b)** Replace y with $4y$.

c) Replace y with $-2y$ and x with $4x$. **d)** Replace y with $y - 4$ and x with $-\frac{1}{4}x$.

Numerical Response

7. The graph of $y = f(x)$ is compressed vertically by a factor of $\frac{1}{2}$ about the x-axis, compressed horizontally by a factor of $\frac{1}{4}$ about the y-axis, and reflected in the y-axis. If the equation of the image is written in the form $y = af(bx)$, the value of $a - b$, to the nearest tenth, is ______ .

Answer Key

1. a) $x \rightarrow \frac{1}{3}x,\ y = f\left(\frac{1}{3}x\right)$ **b)** $y \rightarrow \frac{1}{6}y,\ y = 6f(x)$ **c)** $x \rightarrow \frac{7}{5}x,\ y = f\left(\frac{7}{5}x\right)$

d) $y \rightarrow \frac{3}{2}y,\quad y = \frac{2}{3}f(x)$ **e)** $x \rightarrow -\frac{1}{3}x,\ y = f\left(-\frac{1}{3}x\right)$ **f)** $y \rightarrow -\frac{4}{3}y,\ y = -\frac{3}{4}f(x)$

g) $x \rightarrow -\frac{4}{3}x,\ y = f\left(-\frac{4}{3}x\right)$ **h)** $x \rightarrow \frac{1}{4}x$ and $y \rightarrow \frac{1}{4}y,\ y = 4f\left(\frac{1}{4}x\right)$

i) $x \rightarrow 2x$ and $y \rightarrow -\frac{1}{2}y,\quad y = -2f(2x)$

2. a) $a = 1\quad b = \frac{3}{2}$ **b)** $a = 5\quad b = 1$ **c)** $a = 1\quad b = -\frac{2}{5}$ **d)** $a = \frac{1}{3}\quad b = -10$

3. a) $y = 4f(x) = 4x^2$ **b)** $y = (2x)^2 = 4x^2$ **c)** both transformations result in the same image
d) many possible answers including $f(x) = x, f(x) = x^3, f(x) = x^2 + 1$, etc.

4. **a)** horizontal stretch about the y-axis by a factor of $\frac{1}{k}$

b) vertical stretch about the x-axis by a factor of $\frac{1}{k}$

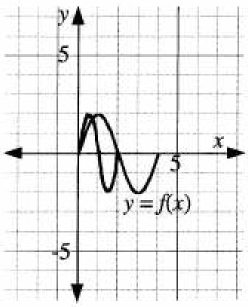
c) vertical translation of k units: up if $k > 0$, down if $k < 0$

d) horizontal translation of k units: right if $k > 0$, left if $k < 0$

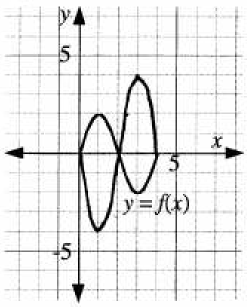
e) vertical stretch about the x-axis by a factor of k

5.

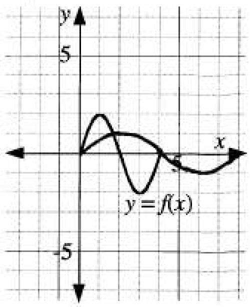
a)



b)

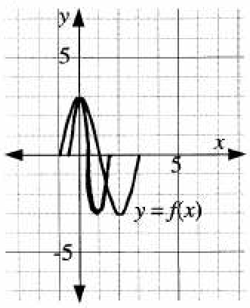


c)

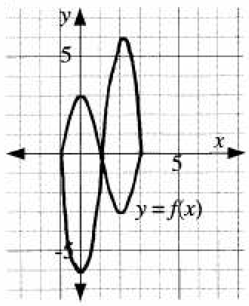


	a)	b)	c)
Domain:	$\{x \mid 0 \le x \le 2, x \in \Re\}$	$\{x \mid 0 \le x \le 4, x \in \Re\}$	$\{x \mid 0 \le x \le 8, x \in \Re\}$
Range:	$\{y \mid -2 \le y \le 2, y \in \Re\}$	$\{y \mid -4 \le y \le 4, y \in \Re\}$	$\{y \mid -1 \le y \le 1, y \in \Re\}$
Invariant Points:	(0, 0)	(0, 0), (2, 0), (4, 0)	(0, 0)

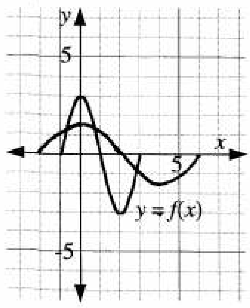
d)



e)



f)



	d)	e)	f)
Domain:	$\left\{x \mid -\frac{1}{2} \le x \le \frac{3}{2}, x \in \Re\right\}$	$\{x \mid -1 \le x \le 3, x \in \Re\}$	$\{x \mid -2 \le x \le 6, x \in \Re\}$
Range:	$\{y \mid -3 \le y \le 3, y \in \Re\}$	$\{y \mid -6 \le y \le 6, y \in \Re\}$	$\left\{y \mid -\frac{3}{2} \le y \le \frac{3}{2}, y \in \Re\right\}$
Invariant Points:	(0, 3)	(−1, 0), (1, 0), (3, 0)	none

6. **a)** horizontal expansion about the y-axis by a factor of 2

b) vertical compression about the x-axis by a factor of $\dfrac{1}{4}$

c) horizontal compression about the y-axis by a factor of $\dfrac{1}{4}$, vertical compression about the x-axis by a factor of $\dfrac{1}{2}$, and a reflection in the x-axis.

d) horizontal expansion about the y-axis by a factor of 4, a reflection in the y-axis, followed by a vertical translation of 4 units up.

7. 4.5