Exponential and Logarithmic Functions Lesson #3:
The Exponential Function

Warm-Up #1 *Exponential Growth*

Mallory, a medical research scientist, discovered a new bacteria culture which could help strengthen a person's immune system. To find the growth rate, she isolated 5 cells of the culture and observed the following growth pattern:

- after one hour there were 10 cells
- after two hours there were 20 cells
- after three hours there were 40 cells

t	$N(t)$
0	5
1	10
2	20
3	40

← a initial value

a) Let t represent the time in hours and $N(t)$ represent the number of cells after t hours. The formula for $N(t)$ as a function of t can be written in the form $N(t) = ab^t$, where a and b are constants and $a, b > 0$. Determine the values of a and b and write the function. pt (1,10) $a=5$

$\frac{10}{5} = \frac{5b^1}{5}$ $b=2$ $N(t) = 5 \cdot (2)^t$

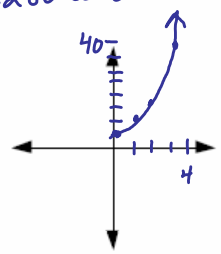
b) Use the formula to determine how many cells there were after 8 hours.

$N(t) = 5(2)^8 = 5(256) = 1280$ cells

c) After 12 hours there are 20 480 cells. How many hours did it take to have half that amount?

double go back 1hr $y = 5 \cdot 2^x$ ∴ at 11hrs

d) Use a graphing calculator to graph the function. A graph of this type represents **exponential growth**.



Window [0, 10, 1]
[0, 50, 5]

Warm-Up #2 *Exponential Decay*

Chernobyl is a city in the former Soviet Union. In April 1986, there was a nuclear accident and the atmosphere was contaminated with quantities of a type of radioactive iodine. At the time of the explosion the atmosphere was contaminated with 32 768 units of radioactive iodine. The following data was recorded for the dissipation of the radioactive iodine:

- after 1 week there was 16 384 units left in the atmosphere
- after 2 weeks there was 8192 units left in the atmosphere
- after 3 weeks there was 4096 units left in the atmosphere

a) Use t to represent the time in weeks and $N(t)$ to represent the number of units of radioactive iodine after t weeks. Complete the following equation of the function which represents the information provided.

$N(t) = 32\,768 \left(\frac{1}{2}\right)^t$

b) Use the formula to determine how many units of radioactive iodine were left after 14 weeks.

$N(t) = 32768\left(\frac{1}{2}\right)^{14} = 2$ units

c) Use a graphing calculator to graph the function using the Window shown. A graph of this type represent **exponential decay**.

```
Xmin=-1
Xmax=8
Xscl=1
Ymin=-30000
Ymax=64000
Yscl=10000
Xres=1
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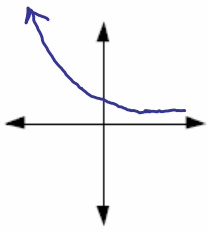
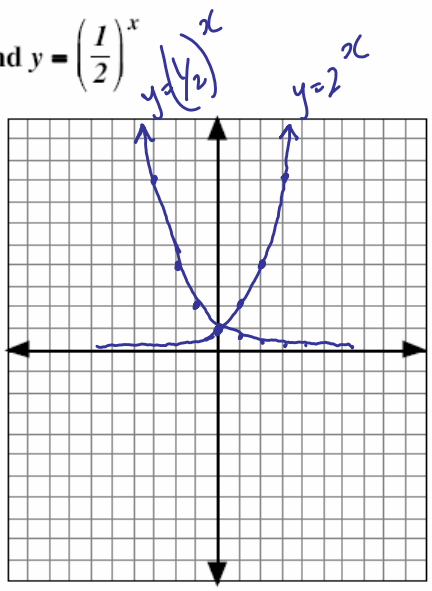


Exponential Function

Warm-Up #1 and Warm-Up #2 are examples of **exponential functions.**
An **exponential function** is a function whose equation is of the form

$y = ab^x$ where $a \neq 0, b > 0, x \in R$

Warm-Up #3 *Comparing the Graphs* $y = 2^x$ *and* $y = \left(\frac{1}{2}\right)^x$



a) State the values of a and b for

$y = 2^x$ and $y = \left(\frac{1}{2}\right)^x$

b) Sketch the graph of the exponential function with equation $y = 2^x, x \in \Re$, using the table of values and grid.

x	-3	-2	-1	0	1	2	3	4
y	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8	16

c) Sketch the graph of the exponential function with equation $y = \left(\frac{1}{2}\right)^x, x \in \Re$, using the table of values and grid.

x	-3	-2	-1	0	1	2	3	4
y	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$

d) Complete the following chart

Equation of Function	*Domain of Function*	*Range of Function*	*x-intercept of Graph*	*y-intercept of Graph*	*Equation(s) of Asymptotes*	
$y = 2^x$	$\{x \mid x \in R\}$	$\{y \mid y > 0, y \in R\}$	None	(0,1)	$y = 0$	(x axis)
$y = \left(\frac{1}{2}\right)^x$	$\{x \mid x \in R\}$	$\{y \mid y > 0, y \in R\}$	None	(0,1)	$y = 0$	(x axis)

e) Complete the following statements using the words "growth" or "decay".

- $f(x) = 2^x$ is an example of a Growth function.
- $f(x) = \left(\frac{1}{2}\right)^x$ is an example of a Decay function.

Warm-Up #4 ***Exploring the Value of b in $y = ab^x$ where $a = 1$***

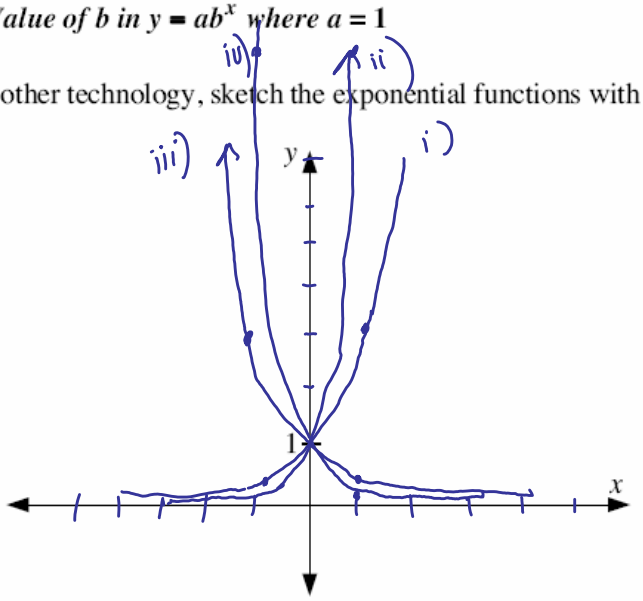
a) By using a graphing calculator or other technology, sketch the exponential functions with equation:

(i) $y = 3^x$ $3^0 = 1$

(ii) $y = 10^x$ $10^0 = 1$

(iii) $y = \left(\frac{1}{3}\right)^x$ $\left(\frac{1}{3}\right)^0 = 1$

(iv) $y = \left(\frac{1}{10}\right)^x$ $\left(\frac{1}{10}\right)^0 = 1$

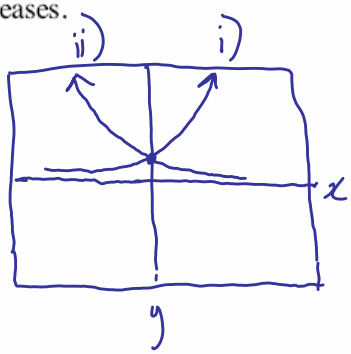


b) The value of b affects the steepness of the graph as x increases. Complete the following.

- When $b > 1$, the curve Rises more sharply as b increases.
- When $0 < b < 1$, the curve Falls more sharply as b decreases.

c) Without using a graphing calculator, make a sketch of the graphs of:

i) $y = 5^x$ **ii)** $y = (0.2)^x$



d) Verify the solution in c) using a graphing calculator

e) State the x-intercept for each of the graphs of the form $y = b^x$.

None

f) State the y-intercept for each of the graphs of the form $y = b^x$.

(0,1) both

g) State the domain for each of the graphs of the form $y = b^x$.

$\{x \mid x \in \mathbb{R}\}$

h) State the range for each of the graphs of the form $y = b^x$.

$\{y \mid y > 0, y \in \mathbb{R}\}$

i) State the equation of the horizontal asymptote for each of the graphs of the form $y = b^x$.

$y = 0$ x axis

Warm-Up #5 ***Exploring the Value of a in $y = ab^x$.***

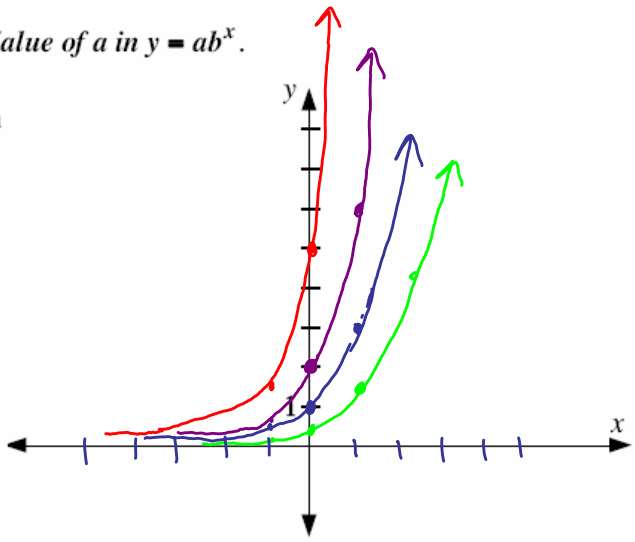
Consider the following functions with equations of the form $y = ab^x$:

i) $y = 3^x$

ii) $y = (2)3^x$

iii) $y = (5)3^x$

iv) $y = (0.5)3^x$



a) Graph the functions using a graphing calculator. Sketch the graphs on the grid .

b) Complete the following table for these equations of the form $y = ab^x$.

	$y = (1)3^x$	$y = (2)3^x$	$y = (5)3^x$	$y = (0.5)3^x$
Value of a	1	2	5	.5
Value of b	3	3	3	3
y-intercept of the graph	1	2	5	.5

c) What is the effect on the graph of changing the value of a?

$y_{int} = a$

d) If $f(x) = 3^x$, write the following in terms of the function f.

i) $y = (3^x)$ $y = f(x)$

ii) $y = (2)3^x$ $y = 2f(x)$

iii) $y = (5)3^x$ $y = 5(f(x)$

iv) $y = (0.5)3^x$ $y = (.5)f(x)$

e) Which transformations are associated with d): ii), iii), iv).

ii) expand vert by factor of 2
iii) expand vert by factor of 5
iv) compress vert by factor of .5

f) What is the effect of the parameter a on the graph of $y = ab^x$

Vertical stretch by factor a

g) State the y-intercept for each of the graphs of the form $y = ab^x$.

$(0, a)$

Characteristics of the Graph of the Exponential Function $f(x) = ab^x$

The following summarizes the basic characteristics of the graph of the exponential function with equation $y = ab^x$.
Use the information from the previous Warm-Ups to complete the following.

- The y-intercept is $(0, a)$. • There is No x-intercept.
- The x-axis is a horizontal asymptote.
- The domain is $\{x \mid x \in \mathbb{R}\}$.
- The range is $\{y \mid y > 0, y \in \mathbb{R}\}$.
- For $a > 0$,
  - when $b > 1$, the function represents a Growth function.
  - when $0 < b < 1$, the function represents a decay function .
- The value of b affects the steepness of the graph as x increases.
  - when $b > 1$, the curve rises sharply as b increases.
  - when $0 < b < 1$, the curve falls sharply as b decreases.
- The value of a affects the vertical stretch of the graph, namely:
  - when $a > 1$, the stretch is an expansion.
  - when $0 < a < 1$, the stretch is a compression
  - when $a < 0$, there is also a reflection on x axis.

Describe how the graph of the second function compares to the graph of the first function.

a) $y = 4^x$

$y = 2(4)^{(x-2)}$

Vert exp by factor of 2
hor trans 2 to the right

b) $y = 2^x$

$y + 4 = -2^{\frac{x}{5}}$ -4 $y = -2^{\left(\frac{1}{5}x\right)} - 4$ $(b = \frac{1}{5})$

hor exp by a factor of 5
Vert tran 4 down
reflection on the x-axis

Explain using transformations why the graph of $y = \left(\frac{1}{3}\right)^x$ is a reflection in the y-axis of the graph of $y = 3^x$.

$y = 3^x$ $y = 3^{-x}$ $\Leftrightarrow$ $y = \left(\frac{1}{3}\right)^x$

Complete Assignment Questions #1 - #8

Assignment

1. State the x and y-intercepts for the graphs of the following:

a) $f(x) = 2^x$ **b)** $f(x) = (2)10^x$ **c)** $f(x) = 2^{(10x)}$ **d)** $y = \left(-\frac{1}{2}\right)\left(\frac{3}{5}\right)^x$

2. **a)** State the domain and range of the function $f(x) = ab^x, a, b > 0, x \in \Re$.

b) Which of the following transformations applied to the graph of $y = ab^x, a, b > 0, x \in \Re$, would result in a change to the **domain** of the function?

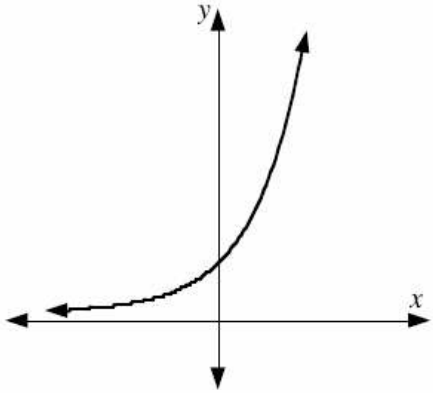
i) horizontal expansion **ii)** vertical compression **iii)** horizontal translation

iv) reflection in the x-axis **v)** reflection in the y-axis **vi)** reflection in the line $y = x$

vii) reciprocal transformation **viii)** absolute value transformation.

c) Which of the above transformations applied to the graph of $y = ab^x, a, b > 0, x \in \Re$, would result in a change to the **range** of the function?

3. The graph of the exponential function with equation $y = b^x$ is shown.



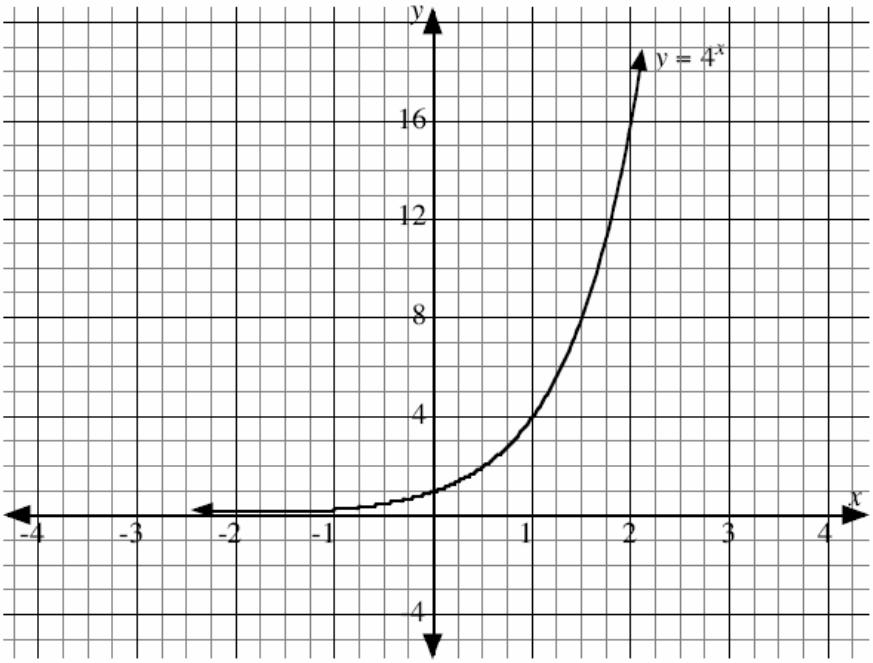
a) Write down the equation of the function after $y = b^x$ is reflected in the y-axis. Sketch this transformation on the grid.

b) Write down the equation of the function after a reciprocal transformation of $y = b^x$. Sketch this transformation on the grid.

c) What do you notice? Explain algebraically.

d) Would your observations in c) be the same if the original function had equation $y = ab^x, a \neq 1$? Explain.

4. The graph of the exponential function with equation $y = 4^x$ is shown.



a) Use the graph to **estimate**, to one decimal place, the solution to the equation $4^x = 12$.

b) Use a graphing calculator to determine, to one decimal place, the solution to the equation $4^x = 12$.

c) On the grid sketch the graph of the function with equation $y = \left(\frac{1}{4}\right)^x$.

d) Without using the grid or a graphing calculator, state the solution to the equation $\left(\frac{1}{4}\right)^x = 12$.

e) Use transformations to sketch the graph of $y = \left(\frac{1}{4}\right)^{x-2} - 4$.

f) State the domain and range of the function $f(x) = \left(\frac{1}{4}\right)^{x-2} - 4$.

g) State the domain and range of the function $f(x) = b^{x-h} + k$.

h) State the equation of the horizontal asymptote of the graph of the function $f(x) = b^{x-h} + k$.

5. Describe how the graph of the second function compares to the graph of the first function.

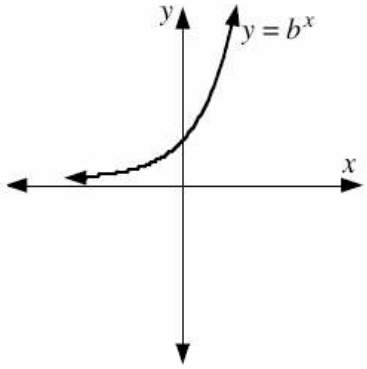
a) $y = 10^x$ $\quad y = 10^{-x} - 3$

b) $y = 2^x$ $\quad y = 5\left(\frac{1}{2}\right)^x$

c) $y = 6^x$ $\quad y = \left(\frac{1}{6}\right)^{-x}$

d) $y = a^x$ $\quad y = a^{\frac{x}{2}}$

6. The graph of $f(x) = b^x$ is shown. Sketch $f^{-1}(x)$.



a) State the domain and range of $f(x)$.

b) State the domain and range of $f^{-1}(x)$.

c) State the asymptotes for $y = f(x)$ and $y = f^{-1}(x)$.

d) Write the equation for the inverse function in the form $x = f(y)$ and try to solve for y. Explain what happens.

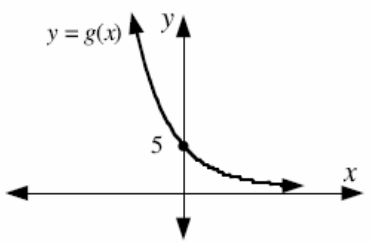
Multiple Choice **7.** Which equation represents an exponential function?

A. $y = 2x^8$ **B.** $y = (-3)^x$ **C.** $y = \frac{3^{x-2}}{2}$ **D.** $y = \frac{1}{3x}$

8. Which equation determines an asymptote of $y = 3^{x-2} + 1$?

A. $y = 3$ **B.** $y = 2$ **C.** $y = 1$ **D.** $y = -1$

Numerical Response **9.** The graph of $g(x) = \frac{1}{f(x)}$ is shown. If $f(x)$ is an exponential function with equation $y = ab^x, a, b > 0$ the value of a to one decimal place is: _____ .



Answer Key

1. **a)** x-intercept: none y-intercept: 1 **b)** x-intercept: none y-intercept: 2 **c)** x-intercept: none y-intercept: 1 **d)** x-intercept: none y-intercept: $-\frac{1}{2}$

2. **a)** Domain: $x \in \Re$ Range: $\{y \mid y > 0, y \in \Re\}$ **b)** vi **c)** iv, vi

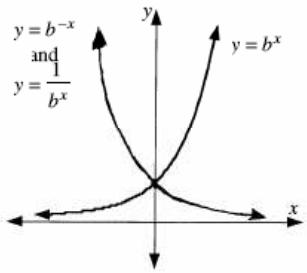
3. **a)** $y = b^{-x}$

b) $y = \frac{1}{b^x}$

c) identical since $b^{-x} = \frac{1}{b^x}$

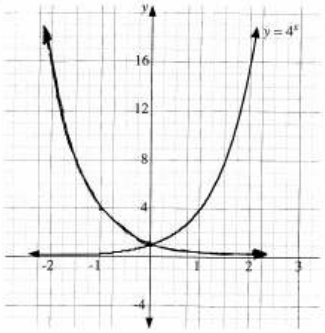
d) No since $ab^{-x} \neq \frac{1}{ab^x}$

3a) sketch → $y = b^{-x}$ and **3b)** sketch → $y = \frac{1}{b^x}$



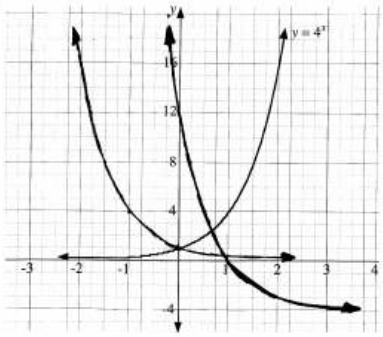
4. **a)** 1.8 **b)** 1.8

c)



d) –1.8

e)



f) Domain: $x \in \Re$ Range: $\{y \mid y > -4, y \in \Re\}$

g) Domain: $x \in \Re$ Range: $\{y \mid y > k, y \in \Re\}$ **h)** $y = k$

5. **a)** reflection in the y-axis, and a vertical translation 3 units down.

b) reflection in the y-axis, and a vertical expansion by a factor of 5

c) identical **d)** horizontal expansion by a factor of 2 about the y-axis

6. **a)** Domain: $x \in \Re$ Range: $\{y \mid y > 0, y \in \Re\}$ **b)** Domain: $\{x \mid x > 0, x \in \Re\}$ Range: $y \in \Re$

c) for $y = f(x)$ the asymptote is $y = 0$ for $y = f^{-1}(x)$ the asymptote is $x = 0$

d) $x = b^y$ don't know how to solve for y until the next lesson

7. C **8.** C **9.** 0.2