Exponential and Logarithmic Functions Lesson #4:
The Logarithmic Function

Warm-Up #1

In Lesson 3, assignment question 6d), we were asked to find the inverse of $y = b^x$ and solve for y. In this lesson we will learn how to do this.

Logarithmic Function

A **logarithmic function** is the inverse of an exponential function. Remember, to find the inverse of a function we must switch x and y and then solve for y. But for the inverse of an exponential function it is difficult to solve for y. For example, to get the inverse of $y = 2^x$ the first step is $x = 2^y$, but it is difficult to solve for y. To solve for y we introduce the logarithmic function as follows:

$$x = 2^y \quad \Rightarrow \quad y = \log_2 x$$

Therefore we write $y = \log_b x$ rather than $x = b^y$ to express the inverse of $y = b^x$.

The **logarithmic function** with base b has the equation

$$y = \log_b x, \quad x > 0, \ x \in R, \ b > 0 \text{ and } b \neq 1.$$

Warm-Up #2

Comparing the Graphs of $y = 2^x$ and $y = \log_2 x$

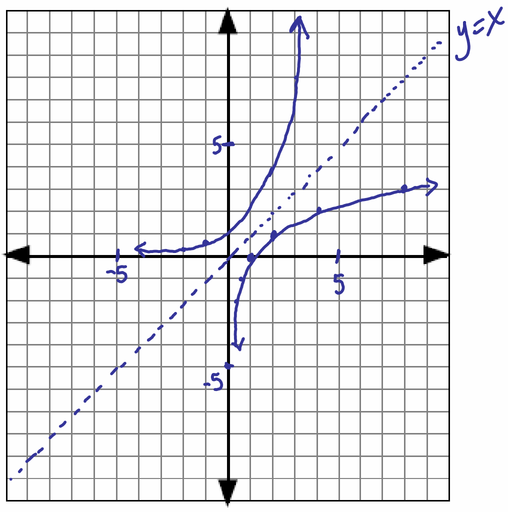
a) Construct the graph of the exponential function $y = 2^x$ and its inverse $y = \log_b x$, (or $x = 2^y$), $x, y \in R$, using the grid and tables of values provided below.

Graph of $y = 2^x$

x	-3	-2	-1	0	1	2	3	4
y	1/8	1/4	1/2	1	2	4	8	16

Graph of $y = \log_2 x$, ($x = 2^y$)

x	1/8	1/4	1/2	1	2	4	8	16
y	-3	-2	-1	0	1	2	3	4



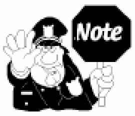
Note $y = 2^x \Longleftrightarrow \log_2 y = x$

b) Complete the table below

Function	Domain	Range	x-intercept	y-intercept	Asymptote
$y = 2^x$	$\{x \mid x \in R\}$	$\{y \mid y > 0, y \in R\}$	None	1	$y = 0$
$y = \log_2 x$	$\{x \mid x > 0, x \in R\}$	$\{y \mid y \in R\}$	1	None	$x = 0$

c) Use the table of values to complete the following.

Logarithmic Form	Exponential Form
$\log_2 8 = 3$	$8 = 2^3$
$\log_2 4 =$	$4 = 2^2$
$\log_2 2 =$	$2 = 2^1$
$\log_2 1 =$	$1 = 2^0$
$\log_2 \frac{1}{2} =$	$\frac{1}{2} = 2^{-1}$
$\log_2 \frac{1}{4} =$	$\frac{1}{4} = 2^{-2}$
$\log_2 \frac{1}{8} =$	$\frac{1}{8} = 2^{-3}$

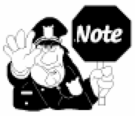


- The logarithms are the exponents in the function $y = 2^x$.
- **Since the logarithmic function $y = \log_b x$ is only defined for positive values of x, the logarithm of a negative number cannot be determined.**

ie $\log(-100)$

Characteristics of the Graph of the Logarithmic Function $y = \log_b x$

- The x intercept is 1.
- There is no y-intercept.
- The y-axis is a vertical asymptote with equation $x = 0$.
- Domain = $\{x \mid x > 0, x \in R\}$.
- Range = $\{y \mid y \in R\}$.
- $y = \log_b x$ is equivalent to $x = b^y$, where $x > 0$ and $b > 0$, $b \neq 1$.
- b is the base of both the logarithmic function and the exponential function.



- **The logarithmic equation $y = \log_b x$ can be expressed in exponential form as $x = b^y$.**
- **The exponential equation $y = b^x$ can be expressed in logarithmic form as $\log_b y = x$.**

Class Ex. #1

Convert each of the following from logarithmic form to exponential form.

a) $\log_7 x = 4$ b) $\log_{10} 1000 = 3$ c) $m = \log_t B$

$x = 7^4$ $10^3 = 1000$ $t^m = B$

d) $\log_x a = b^d$ e) $5 = 4\log_b 6$

$x^{b^d} = a$ $\frac{5}{4} = \log_b 6$ $b^{\frac{5}{4}} = 6$

Class Ex. #2

Convert each of the following from exponential form to logarithmic form.

a) $4^3 = 64$ b) $2^{-3} = \frac{1}{8}$ c) $e^d = f$

$3 = \log_4(64)$ $-3 = \log_2\left(\frac{1}{8}\right)$ $d = \log_e(f)$

d) $x^{2y} = 5$ e) $a = (2x + 4)^{-1}$

$2y = \log_x 5$ $-1 = \log_{(2x+4)}(a)$

Class Ex. #3

Solve for y.

a) $\log_3 81 = y$ b) $y = \log_5\sqrt{125}$ c) $y = 2\log_8 512$

$81 = 3^y$, $3^4 = 3^y$, $y = 4$

$5^y = \sqrt{125}$, $5^y = 5^{\frac{3}{2}}$, $y = \frac{3}{2}$

$\frac{y}{2} = \log_8 512$, $8^{\frac{y}{2}} = 512$, $(2^3)^{\frac{y}{2}} = 2^9$, $\left(\frac{2}{3}\right)\frac{3y}{2} = 9\left(\frac{2}{3}\right)$, $y = \frac{18}{3} = 6$

Class Ex. #4

a) What is the value of $\log_b 1$? Explain.

$\log_b 1 = y$, $b^y = 1$, $b^y = b^0$, $y = 0$, So $\log_b 1 = 0$

b) What is the value of $\log_b b$? Explain.

$\log_b b = y$, $b^y = b^1$, $y = 1$, So $\log_b b = 1$

Class Ex. #5

Evaluate.

a) $\log_4 64 = y$ b) $\log_2\left(\frac{1}{32}\right) = y$ c) $5^{(\log_5 25)} = y$ d) $a^{(\log_a a)} = y$

$4^y = 64$, $4^y = 4^3$, $y = 3$

$2^y = \frac{1}{32}$, $2^y = 2^{-5}$, $y = -5$

$5^y = 25$, $5^y = 5^2$, $y = 2$, $5^2 = 25$

$a^y = a^1$, $y = 1$, $a^1 = a$

Class Ex. #6

Find the inverse of the following equations. Answer in the form $y =$ ________ . i) interchange x for y

ii) Solve for y

a) $y = \log_3 x$ i) $x = \log_3 y$ ii) $3^x = y$

b) $y = 8^x$ i) $x = 8^y$ ii) $8^x = y$

Complete Assignment Questions #1 - #9

Warm-Up #3

We have seen how to change forms between the exponential form $y = b^x$ and the logarithmic form $\log_b y = x$.

We now consider how to write the exponential form $y = ab^x$ in logarithmic form.

This can be done using the following procedure:

1. Write the exponential form $y = ab^x$ as $\frac{y}{a} = b^x$.
2. Change $\frac{y}{a} = b^x$ to logarithmic form.

The logarithmic form of $y = ab^x \left(\text{or } \frac{y}{a} = b^x \right)$ is $x = \log_b \left(\frac{y}{a} \right)$.

Read Base "b" to exponent "x" equals $\left(\frac{y}{a} \right)$

Class Ex. #7

Change each of the following from exponential form to logarithmic form.

a) $y = 2(3^x)$ $\quad \frac{y}{2} = 3^x \quad x = \log_3 \left(\frac{y}{2} \right)$

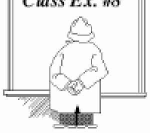
b) $h = 7(4)^k$ $\quad \frac{h}{7} = (4)^k \quad k = \log_4 \left(\frac{h}{7} \right)$

c) $5y = 30^x$ $\quad x = \log_{30} (5y)$

d) $\frac{2}{3} y = \frac{3}{2}(10)^x$ $\quad \frac{2y}{3} = 10^x \quad x = \log_{10} \left(\frac{2y}{3} \right)$

e) $t = r(s)^p$ $\quad \frac{t}{r} = s^p \quad p = \log_s \left(\frac{t}{r} \right)$

Class Ex. #8



Change each of the following from logarithmic form to exponential form $y = ab^x$.

a) $\log_7 \left(\frac{y}{3} \right) = x$ $\quad (3)\frac{y}{3} = (3)7^x \quad y = 3 \cdot (7)^x$

b) $\log_{10} \left(\frac{y}{4} \right) = x$ $\quad (4)\frac{y}{4} = (4)\,10^x \quad y = 4 \cdot (10)^x$

c) $\log_5 (7y) = x$ $\quad \frac{7y}{7} = \frac{5^x}{7} \quad y = \frac{1}{7} \cdot (5)^x$

d) $\log_e \left(\frac{y}{5} \right) = x$ $\quad (5)\frac{y}{5} = (5)e^x \quad y = 5 \cdot (e)^x$

Complete Assignment Questions #10 - #16

Assignment

1. Complete the following from the graphs of $y = b^x$ and $y = \log_b x$, $b > 0$.

Function	Domain	Range	x-intercept	y-intercept	Asymptote
$y = b^x$					
$y = \log_b x$					

2. Why does x have to be greater than zero in the domain of $y = \log_b x$, and not in $y = b^x$, $b > 0$?

3. Express each of the following in logarithmic form.

a) $5^2 = 25$ **b)** $3^0 = 1$ **c)** $2^{-4} = \frac{1}{16}$

d) $\left(\frac{1}{2}\right)^4 = \frac{1}{16}$ **e)** $b^d = e$

4. Express each of the following in exponential form.

a) $\log_3 9 = 2$ **b)** $\log_5 625 = 4$ **c)** $\log_4 \frac{1}{4} = -1$

d) $\log_a f = i$ **e)** $\log_{10} 0.001 = -3$

5. Evaluate.

a) $\log_4 64$ **b)** $\log_5 \sqrt{5}$ **c)** $\log_7 49$

d) $\log_{10} 0.001$ **e)** $\log_8 8^{-4}$ **f)** $\log_2 \sqrt{\frac{1}{512}}$

g) $\log_b 1$ **h)** $\log_c c$ **i)** $\log_x x^z$

6. Complete the following table:

Logarithmic Form	Exponential Form	Value of x
$\log_4 x = 2$		
	$7 = 49^x$	
$\log_x\left(\frac{1}{64}\right) = -3$		
	$x + 2 = 4^2$	
$\log_{32} x = \frac{1}{5}$		
	$\frac{1}{2} = 16^x$	

7. Is $y = \log_3 x$ the logarithmic form of $y = 3^x$? Explain your answer.

8. Solve for x.

a) $\log_x 125 = 3$ **b)** $\log_{125} 5 = x$ **c)** $\log_4 x = -8$

9. Find the inverse of the following equations. Answer in the form $y =$ ________ .

a) $y = 3^x$ **b)** $y = \log_4 x$ **c)** $y = 3x^2 + 2$

d) $y = \log_3 x$ **e)** $y = 20^x$ **f)** $x = 20^y$

10. Change each of the following from exponential form to logarithmic form.

a) $y = 3(2)^x$ **b)** $y = 10(3)^x$ **c)** $7y = 30^x$

d) $y = \frac{5}{6}(10)^x$ **e)** $a = b(c)^d$ **f)** $2y = 7\left(\frac{3}{4}\right)^x$

11. Change each of the following from logarithmic form to exponential form $y = ab^x$.

a) $\log_8\left(\frac{y}{9}\right) = x$ **b)** $\log_{20}(6y) = x$ **c)** $\log_e\left(\frac{y}{5}\right) = x$ **d)** $\log_{10}(0.5y) = x$

12. Solve the following equations for y.

a) $3 = \log_2\left(\frac{y}{4}\right)$ **b)** $\log_2\left(\frac{y}{5}\right) = -3$ **c)** $2 = \log_4 32y$

Multiple Choice

13. If $\log_4(4096x) = 64$, then the value of x is

A. $4^{\frac{32}{3}}$

B. 4^{58}

C. 4^{6}

D. 4^{32}

14. If $\log_2 x = 3$ and $\log_2 t = x$, then t equals

A. 64

B. 128

C. 256

D. 512

Numerical Response

15. If $\log_b 81 = \frac{2}{3}$, then the value of b is to the nearest whole number is: ______ .

Numerical Response

16. If $\log_a b = 4.5$ and $\log_a c = 3.7$, then the value of $\log_a\left(\frac{b}{c}\right)$ to the nearest tenth is: ______ .

Answer Key

	Function	Domain	Range	x-intercept	y-intercept	Asymptote
1	$y = b^x$	$x \in \Re$	$\{y \mid y > 0, y \in \Re\}$	none	1	$y = 0$
	$y = \log_b x$	$\{x \mid x > 0, x \in \Re\}$	$y \in \Re$	1	none	$x = 0$

2. $y = \log_b x \Rightarrow x = b^y$; b^y must be greater than zero, so x must be greater than zero.
$y = b^x$ can be determined for all values of x positive, negative, or zero.

3. **a)** $\log_5 25 = 2$ **b)** $\log_3 1 = 0$ **c)** $\log_2\left(\frac{1}{16}\right) = -4$ **d)** $\log_{\frac{1}{2}}\left(\frac{1}{16}\right) = 4$ **e)** $\log_b e = d$

4. **a)** $3^2 = 9$ **b)** $5^4 = 625$ **c)** $4^{-1} = \frac{1}{4}$ **d)** $a^i = f$ **e)** $10^{-3} = 0.001$

5. **a)** 3 **b)** $\frac{1}{2}$ **c)** 2 **d)** -3 **e)** -4 **f)** $\frac{-9}{2}$ **g)** 0 **h)** 1 **i)** z

6.

Logarithmic Form	Exponential Form	Value of x
$\log_4 x = 2$	$x = 4^2$	$x = 16$
$\log_{49} 7 = x$	$7 = 49^x$	$x = \frac{1}{2}$
$\log_x\left(\frac{1}{64}\right) = -3$	$\frac{1}{64} = x^{-3}$	$x = 4$
$\log_4(x + 2) = 2$	$x + 2 = 4^2$	$x = 14$
$\log_{32} x = \frac{1}{5}$	$x = 32^{\frac{1}{5}}$	$x = 2$
$\log_{16}\left(\frac{1}{2}\right) = x$	$\frac{1}{2} = 16^x$	$x = -\frac{1}{4}$

7. No, it is the inverse. The log form of $y = 3^x$ is $x = \log_3 y$.

8. **a)** 5 **b)** $\frac{1}{3}$ **c)** 0.0000153

9. **a)** $y = \log_3 x$ **b)** $y = 4^x$ **c)** $y = \pm\sqrt{\frac{x-2}{3}}$
d) $y = 3^x$ **e)** $y = \log_{20} x$ **f)** $y = 20^x$

10. **a)** $\log_2\left(\frac{y}{3}\right) = x$ **b)** $\log_3\left(\frac{y}{10}\right) = x$ **c)** $\log_{30}(7y) = x$
d) $\log_{10}\left(\frac{6y}{5}\right) = x$ **e)** $\log_c\left(\frac{a}{b}\right) = d$ **f)** $\log_{\frac{3}{4}}\left(\frac{2y}{7}\right) = x$

11. **a)** $y = 9(8)^x$ **b)** $y = \frac{1}{6}(20)^x$ **c)** $y = 5(e)^x$ **d)** $y = 2(10)^x$

12. **a)** 32 **b)** $\frac{5}{8}$ **c)** $\frac{1}{2}$ **13.** B **14.** C **15.** 729 **16.** 0.8