Exponential & Logarithmic Functions Lesson #5:
Changing the Base of Logarithms

Warm-Up #1

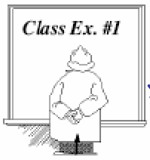
In the previous lesson we discussed logarithms with many different bases. In order to do numerical calculations on a calculator, we use two specific bases - base 10 and base e.

Common Logarithms

Common logarithms are logarithms in base 10, eg. $\log_{10}1000$. These logarithms are in such common use that when a base is not given the logarithm is understood to be in base ten. For instance,

$\log_{10}1000$ is often written as $\log 1000$

On a graphing calculator, this can be evaluated using the LOG key.



Evaluate each of the following logarithms manually and by calculator.

a) $\log 1000$

i) Manually $\log_{10} 10^3 = 3$ ii) Calculator $\log 1000 = 3$

b) $\log \sqrt[3]{1000}$

i) manually $\log_{10} (10^3)^{\frac{1}{3}}$, $\log_{10} 10^1 = 1$ ii) Calculator $\log(\sqrt[3]{1000}) = 1$

Natural Logarithms

Natural logarithms are logarithms in base e eg. $\log_e 15$. e is an irrational number, the value of which will be determined in Warm-Up #2.

$\log_e 15$ is often written as $\ln 15$

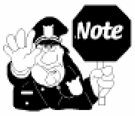
On a graphing calculator, this can be evaluated using the LN key.

Evaluate the following logarithms to one decimal place where necessary.

a) $\ln 5$ Calculator $= 1.6$

b) $\log_e 5$ $= \ln 5 = 1.6$

c) $\ln e$ $= 1$



Warm-Up #2 is not required for this course, but may give students a greater understanding in preparation for higher level math courses.

Warm-Up #2 *Approximating the Value of e*

The formal definition of the irrational number e is the limit as x approaches infinity of the function $f(x) = \left(1 + \frac{1}{x}\right)^x$. Complete the following table to determine the value of this function as x gets very large. Use the TABLE feature of a graphing calculator and work to 4 decimal places.

x	10	100	1000	10 000	100 000	1 000 000
$\left(1 + \frac{1}{x}\right)^x$	2.5937	2.7048	2.7169	2.7181	2.7183	2.7183

Estimate for e is 2.7183.

A more accurate estimate can be determined by pressing the e key on a graphing calculator.

The value of e to 9 decimal places is 2.718281828.

Warm-Up #3

a) Evaluate $\log_5 25$

$\log_5 5^2$ or $\log_5 25 = y$

$= 2$ $\quad 5^y = 25$

$5^y = 5^2 \quad y = 2$

b) Try to evaluate $\log_5 50$. What problem do you encounter?

$\log_5 50 = y$

$5^y = 50$

Base 5 to an exponent (whole number) does not equate to 50

At the moment we are unable to evaluate $\log_5 50$, but by converting to common logarithms, or natural logarithms, we can use a calculator to determine the value of $\log_5 50$. The method for converting from one base to another is discussed in Warm-Up #4

Warm-Up #4 — *Change of Base*

a) Evaluate.

i) $\log_5 25$ = y $5^y = 5^2$ y = 2

ii) $\dfrac{\log 25}{\log 5}$ Calculator   = 2

iii) $\dfrac{\log_e 25}{\log_e 5}$ ($\dfrac{\ln 25}{\ln 5}$) = $\dfrac{3.2188}{1.609}$ = 2

b) Evaluate.

i) $\log_3 243$ = y $3^y = 3^5$ y = 5

ii) $\dfrac{\log 243}{\log 3}$ Calculator   = 5

iii) $\dfrac{\log_e 243}{\log_e 3}$ ($\dfrac{\ln 243}{\ln 3}$) = 5

c) Write $\log_2 64$ in a form which can be evaluated using a calculator.

= $\dfrac{\log 64}{\log 2}$ Input log64 ÷ log2 Enter = 6

Change of Base Identity

Change of Base Identity

$$\log_b c = \frac{\log_a c}{\log_a b}$$

This formula is NOT on the formula sheet

We have seen in Warm-Up #4 that the above identity is true for converting logarithms to base 10 or base *e*. In fact it holds true for converting logarithms to any base. The example below supports this.

i) Evaluate $\log_4 1024$

= $\dfrac{\log 1024}{\log 4}$ = 5

ii) Evaluate $\dfrac{\log_2 1024}{\log_2 4}$ = $\dfrac{\log 1024 \div \log 2}{\log 4 \div \log 2}$ = $\dfrac{10}{2}$ = 5

We will be able to prove this identity in a later lesson when we have developed an understanding of the laws of logarithms.

Class Ex. #3

Evaluate the following logarithms to the nearest hundredth by changing the base.

a) $\log_5 221$

= $\dfrac{\log 221}{\log 5}$ = 3.35

b) $\log_2 \dfrac{1}{1000}$

= $\dfrac{\log(1 \div 1000)}{\log 2}$ = −9.97

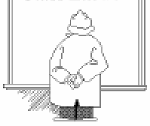
c) $3\log_7 512$

= $3\left(\dfrac{\log 512}{\log 7}\right)$

= 3(3.2058....)

= 9.62

Class Ex. #4



Convert the following logarithms to the base indicated.

a) $\log_6 216$ to base 3

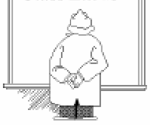
$= \dfrac{\log_3 216}{\log_3 6}$

b) $\log 300$ to base 5

10

$= \dfrac{\log_5 300}{\log_5 10}$

Class Ex. #5



Find the exact value of the following.

a) $\log_3 \dfrac{1}{729}$

$= \dfrac{\log(1 \div 729)}{\log 3}$

$= -6$

b) $2\log_8 512$

$= 2\left(\dfrac{\log 512}{\log 8}\right)$

$= 2(3) = 6$

c) $-\log_7\left(\dfrac{1}{343}\right)$

$= -1\left(\dfrac{\log(1 \div 343)}{\log 7}\right)$

$= -1(-3) = 3$

d) $5^{(\log_5 25)}$

i) $\dfrac{\log 25}{\log 5} = 2$

ii) $5^2 = 25$

e) $\log_2 \sqrt{\dfrac{1}{1024}}$

$= \dfrac{\log\left(\sqrt{1 \div 1024}\right)}{\log 2}$

$= 1/2$

f) $\log_7 49^{-5}$

$= \dfrac{\log(49^{-5})}{\log 7}$

$= -10$

Complete Assignment Questions #1 - #10

Assignment

1. Evaluate each of the following logarithms.

a) $\log 100$ **b)** $\log 10^6$ **c)** $\log\sqrt{10}$ **d)** $\log 0.01$

2. Evaluate the following logarithms to the nearest tenth.

a) $\ln 20$ **b)** $\log_e 8$ **c)** $\ln e^2$

3. Convert the following logarithms to the base indicated.

a) $\log_8 35$ to base 7 **b)** $\log \dfrac{1}{2}$ to base 6 **c)** $\log_3 50$ to base e

4. Evaluate using the change of base identity to the nearest hundredth:

a) $\log_5 17$ **b)** $\log_{0.5} 5.9$ **c)** $\dfrac{1}{\log_5 3}$ **d)** $-2\log_{12} 6$ **e)** $\log_8 8$

5. Evaluate each expression:

a) $4^{\log_4 4}$ **b)** $10^{\log_{10} 1000}$

6. Describe how to graph $y = \log_3 x$ using a graphing calculator. Sketch the graph and determine the x-intercept.

Multiple Choice **7.** Which of the following has a negative value?

A. $-\log_4(0.1)$ **B.** $\log_4\left(\dfrac{5}{2}\right)$ **C.** $\log_{\frac{1}{2}}\left(\dfrac{2}{3}\right)$ **D.** $\log_4\left(\dfrac{2}{3}\right)$

Numerical Response **8.** The value of the expression $\log_{\sqrt{2}} 8 + 2\log_9 3$ to the nearest tenth is _____ .

9. Given the equation $\log_7 x = \log_4 60$, the value of x to the nearest whole number is _____ .

10. If $\log_x 27 = \log_{12} 3$, the value of x to the nearest whole number is _____ .

Answer Key

1. **a)** 2 **b)** 6 **c)** $\frac{1}{2}$ **d)** –2

2. **a)** 3.0 **b)** 2.1 **c)** 2.0

3. **a)** $\frac{\log_7 35}{\log_7 8}$ **b)** $\frac{\log_6\left(\frac{1}{2}\right)}{\log_6 10}$ **c)** $\frac{\log_e 50}{\log_e 3}$

4. **a)** 1.76 **b)** –2.56 **c)** 1.46 **d)** –1.44 **e)** 1.00

5. **a)** 4 **b)** 1000

6. Graph $y = \frac{\log x}{\log 3}$, x-intercept is 1

7. D

8. 7.0

9. 313

10. 1728