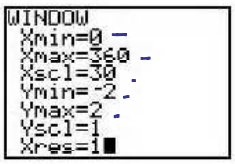
Trigonometry - Functions and Graphs Lesson #2:
Reference Angles and the CAST Rule

Warm-Up #1

a) Use a calculator to find the value of sin 30° and sin 150° and compare the answers.

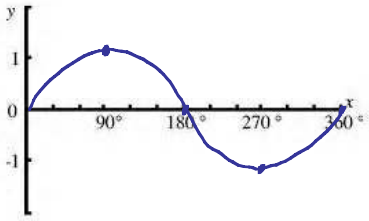
b) Compare the value of cos 40° and cos 140° (to four decimal places).

- To show why the above answers are the same, we will construct and analyze the graphs of $y = \sin x$ and $y = \cos x$ from 0° to 360°.
- Use the following mode and window settings.



Warm-Up #2 *Exploring the Graph of* $y = \sin x$

a) Sketch the graph of $y = \sin x$, $0° \le x \le 360°$, on the grid.



b) Use the trace feature to complete the table below to four decimal places where necessary.

Press [Trace], enter the value of x,

then press [Enter] to find the value of y.

x (angle in degrees)	y (sine ratio)
0°	0
30°	0.5
45°	0.7071
60°	0.8660
90°	1

x (angle in degrees)	y (sine ratio)
120°	0.8660
135°	0.7071
150°	0.5
180°	0

x (angle in degrees)	y (sine ratio)
210°	−0.5
225°	−0.7071
240°	−0.8660
270°	−1

x (angle in degrees)	y (sine ratio)
300°	−0.8660
315°	−0.7071
330°	−0.5
360°	0

c) Without using a calculator, state two angles (not in the table) which have:

i) the same positive value for the sine ratio. 40 and 140

ii) the same negative value for the sine ratio. 200 and 340

Graphing primary and secondary trigonometric functions will be studied in more detail later on in this unit.

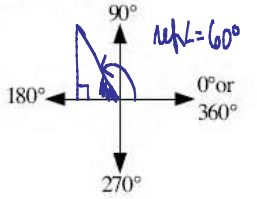
Reference Angles

In order to investigate pairs of angles with identical trigonometric ratios, we introduce the concept of a **reference angle**. A **reference angle** is the acute angle formed between the terminal arm of the rotation angle and the x-axis.

Class Ex. #1

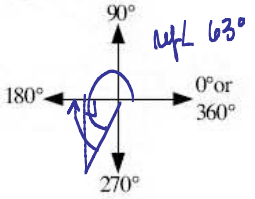
In each case, sketch the rotation angle and state the reference angle.

a) 120°



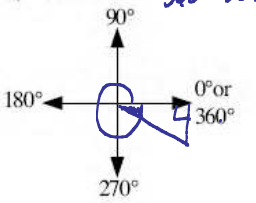
180 - 120
ref ∠ = 60°

b) 243°



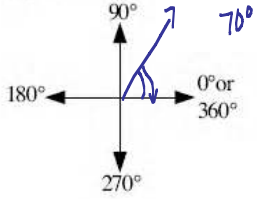
243 - 180
ref ∠ 63°

c) 337°



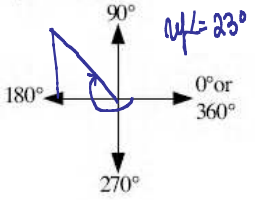
360 - 337 = 23°
ref ∠

d) 70°



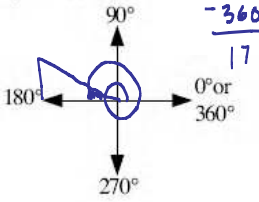
70 - 0
70°

e) –203°



-203 + 180
ref ∠ = 23°

f) 537°



537
-360
177
180
-177
ref ∠ = 3°

Class Ex. #2

Determine the positive rotation angle, x, $0° \le x < 360°$, given the reference angle and the quadrant.

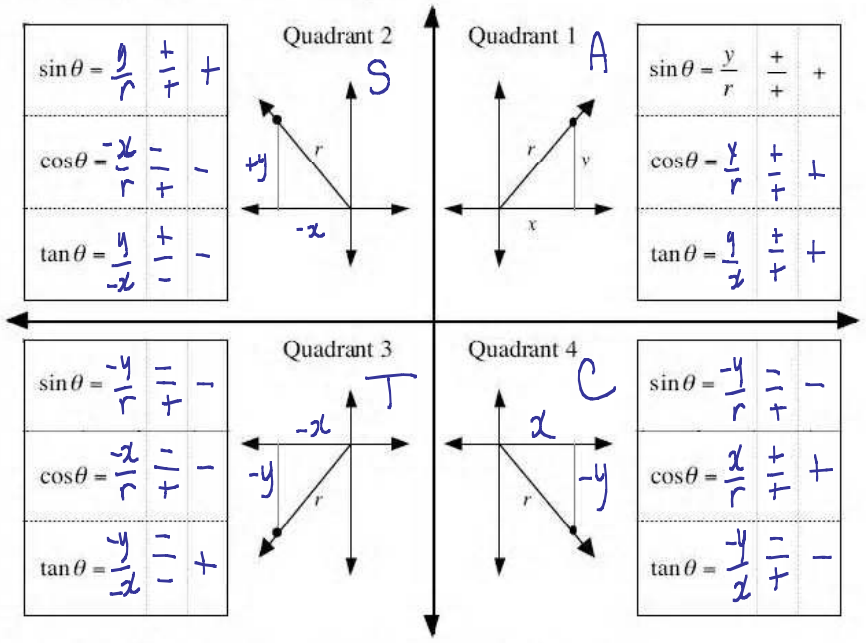
Reference Angle	Quadrant	Sketch	Rotation Angle
25°	2		180 - 25 155°
60°	4		360 - 60 300°
8°	3		180 + 8 188°
39°	1		0 + 39 = 39°
90°	between 3 and 4		180 + 90 270°

Complete Assignment Question #1 - #4

Warm-Up #3 — *Determining the Sign of a Trigonometric Ratio*

a) In each quadrant draw the rotation angle θ in standard position.

b) Complete the chart to determine the sign of each ratio.
Use sin θ in Quadrant 1 as an example.



Quadrant 2			Quadrant 1		
$\sin\theta = \frac{y}{r}$	$\frac{+}{+}$	+	$\sin\theta = \frac{y}{r}$	$\frac{+}{+}$	+
$\cos\theta = \frac{-x}{r}$	$\frac{-}{+}$	−	$\cos\theta = \frac{x}{r}$	$\frac{+}{+}$	+
$\tan\theta = \frac{y}{-x}$	$\frac{+}{-}$	−	$\tan\theta = \frac{y}{x}$	$\frac{+}{+}$	+

Quadrant 3			Quadrant 4		
$\sin\theta = \frac{-y}{r}$	$\frac{-}{+}$	−	$\sin\theta = \frac{-y}{r}$	$\frac{-}{+}$	−
$\cos\theta = \frac{-x}{r}$	$\frac{-}{+}$	−	$\cos\theta = \frac{x}{r}$	$\frac{+}{+}$	+
$\tan\theta = \frac{-y}{-x}$	$\frac{-}{-}$	+	$\tan\theta = \frac{-y}{x}$	$\frac{-}{+}$	−

c) Complete the following statements using the results from a).

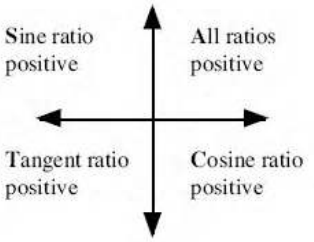
i) Sine ratios have **positive** values in quadrants 1 and 2.
ii) Cosine ratios have **positive** values in quadrants 1 and 4.
iii) Tangent ratios have **positive** values in quadrants 1 and 3.
iv) Sine ratios have **negative** values in quadrants 3 and 4.
v) Cosine ratios have **negative** values in quadrants 2 and 3.
vi) Tangent ratios have **negative** values in quadrants 2 and 4.

CAST Rule

The results can be memorized by:

- the **CAST** rule or
- by remembering to "**A**dd **S**ugar **T**o **C**offee"

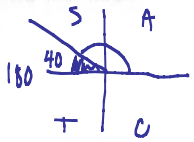
The reciprocal trigonometric ratios follow the same framework as their corresponding primary ratio.

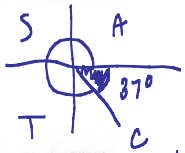


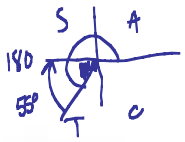
The trigonometric ratios of <u>any</u> angle can be written as the same function of a positive acute angle called the reference angle with the sign of the ratio being determined by the CAST rule.

Rewrite as the same trigonometric function of a positive acute angle.

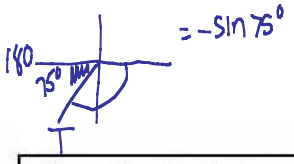
a) $\sin 140°$ $= +\sin 40$ **b)** $\tan 323°$ $= -\tan 37°$ **c)** $\cos 235°$ $= -\cos 55°$



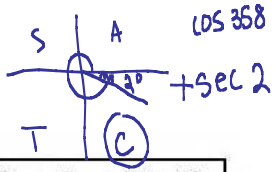




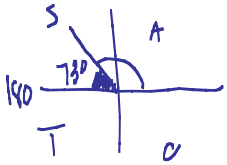
d) $\sin(-105)°$ **e)** $\sec 358°$ $= \frac{1}{\cos 358}$ **f)** $\cot 107°$ $\frac{1}{\tan 107} = -\cot 73°$



$= -\sin 75°$



$+\sec 2$

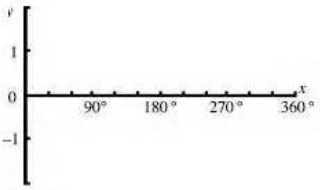


Complete Assignment Questions #5 - #8

Assignment

1. Use a graphing calculator with the settings for Warm-Up #1 to answer the following.

a) Sketch the graph of $y = \cos x, 0° \le x \le 360°$, on the grid.



b) Complete the table to four decimal places where necessary.

x (angle in degrees)	y (cosine ratio)
0°	
30°	
45°	
60°	
90°	

x (angle in degrees)	y (cosine ratio)
120°	
135°	
150°	
180°	

c) Without using a calculator, state two angles (not in the table) which have:

x (angle in degrees)	y (cosine ratio)
210°	
225°	
240°	
270°	

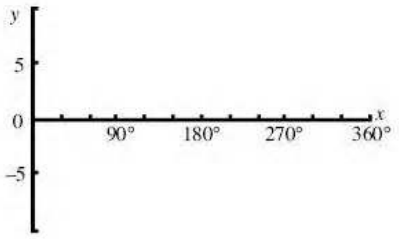
x (angle in degrees)	y (cosine ratio)
300°	
315°	
330°	
360°	

i) the same positive value for the cosine ratio.

ii) the same negative value for the cosine ratio.

2. Use a graphing calculator in degree mode with window setting x: [0, 360, 45] and y: [−10, 10, 1] to answer the following.

a) Sketch the graph of $y = \tan x, 0° \le x \le 360°$, on the grid.



b) Complete the table to four decimal places where necessary.

x (angle in degrees)	y (tan ratio)
0°	
30°	
45°	
75°	
85°	
89°	
89.99°	
90°	

x (angle in degrees)	y (tan ratio)
90.01°	
91°	
95°	
105°	
120°	
135°	
150°	
180°	

c) State the equations of the asymptotes of the graph of $y = \tan x$, $0° \le x \le 360°$.

x (angle in degrees)	y (tan ratio)
200°	
210°	
225°	
255°	
265°	
269°	
269.99°	
270°	

x (angle in degrees)	y (tan ratio)
270.01°	
271°	
275°	
285°	
300°	
315°	
330°	
360°	

d) Without using a calculator, state two angles (not in the table) which have:

i) the same positive value for the tangent ratio.

ii) the same negative value for the tangent ratio.

3. Find the reference angle for the following rotation angles.

a) 135° **b)** 296° **c)** 237° **d)** –25°

e) –245° **f)** 820° **g)** 180° **h)** –270°

i) 0° **j)** 90° **k)** 270° **l)** 360°

4. Complete the following tables given the reference angle and the quadrant.

Reference Angle	Quadrant	Sketch	Rotation Angle
30°	2		
45°	4		
60°	1		
25°	3		
15°	4		
36°	3		

Reference Angle	Quadrant	Sketch	Rotation Angle
30°	1		
30°	4		
4°	3		
89°	2		
0°	between 2 and 3		
90°	between 1 and 2		

5. Complete the following statements.

a) Secant ratios have positive values in quadrants _____ and _____ .

b) Cosecant ratios have positive values in quadrants _____ and _____ .

c) Cotangent ratios have positive values in quadrants _____ and _____ .

6. In which quadrant(s) does the terminal arm lie if:

a) $\sin\theta$ is negative?

b) $\sec\theta$ is positive?

c) $\csc\theta$ and $\tan\theta$ are both negative?

d) $\cot\theta$ is positive and $\csc\theta$ is negative?

7. Rewrite as the same trigonometric function of a positive acute angle.

a) $\sin 205° =$

b) $\cot 193° =$

c) $\sec 107° =$

d) $\csc (-380°) =$

e) $\cos 451° =$

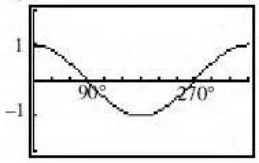
f) $\tan (-30°) =$

Multiple Choice **8.** Which of the following is $\tan (-105°)$ expressed as the same trigonometric function of a positive acute angle?

A. $\tan 255°$
B. $\tan 15°$
C. $\tan 75°$
D. $-\tan 75°$

Answer Key

1. a)



b)

x (angle in degrees)	y (cosine ratio)
0°	1
30°	0.8660
45°	0.7071
60°	0.5
90°	0

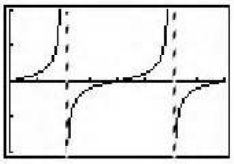
x (angle in degrees)	y (cosine ratio)
120°	–0.5
135°	–0.7071
150°	–0.8660
180°	–1

x (angle in degrees)	y (cosine ratio)
210°	–0.8660
225°	-0.7071
240°	–0.5
270°	0

x (angle in degrees)	y (cosine ratio)
300°	0.5
315°	0.7071
330°	0.8660
360°	1

c) answers may vary
i) 70°, 290°
ii) 130°, 230°

2. a)



b)

x (angle in degrees)	y (tan ratio)
0°	0
30°	0.5774
45°	1
75°	3.7321
85°	11.4301
89°	57.2900
89.99°	5729.5779
90°	undefined

x (angle in degrees)	y (tan ratio)
90.01°	–5729.5779
91°	–57.2900
95°	–11.4301
105°	–3.7321
120°	–1.7321
135°	–1
150°	–0.5774
180°	0

x (angle in degrees)	y (tan ratio)
200°	0.3640
210°	0.5774
225°	1
255°	3.7321
265°	11.4301
269°	57.2900
269.99°	5729.5779
270°	undefined

x (angle in degrees)	y (tan ratio)
270.01°	–5729.5779
271°	–57.2900
275°	–11.4301
285°	–3.7321
300°	–1.7321
315°	–1
330°	–0.5774
360°	0

c) $x = 90°, x = 270°$

d) answers may vary
i) 70°, 250°
ii) 130°, 310°

3. a) 45° **b)** 64° **c)** 57° **d)** 25°
e) 65° **f)** 80° **g)** 0° **h)** 90°
i) 0° **j)** 90° **k)** 90° **l)** 0°

4.

Reference Angle	Quadrant	Sketch	Rotation Angle
30°	2		150°
45°	4		315°
60°	1		60°
25°	3		205°
15°	4		345°
36°	3		216°

Reference Angle	Quadrant	Sketch	Rotation Angle
30°	1		30°
30°	4		330°
4°	3		184°
89°	2		91°
0°	between 2 and 3		180
90°	between 1 and 2		90°

5. a) 1 and 4 **b)** 1 and 2 **c)** 1 and 3

6. a) quadrant 3 and 4
b) quadrant 1 and 4
c) quadrant 4
d) quadrant 3

7. a) –sin 25° **b)** cot 13° **c)** –sec 73° **d)** –csc 20° **e)** –cos 89° **f)** –tan 30°

8. C