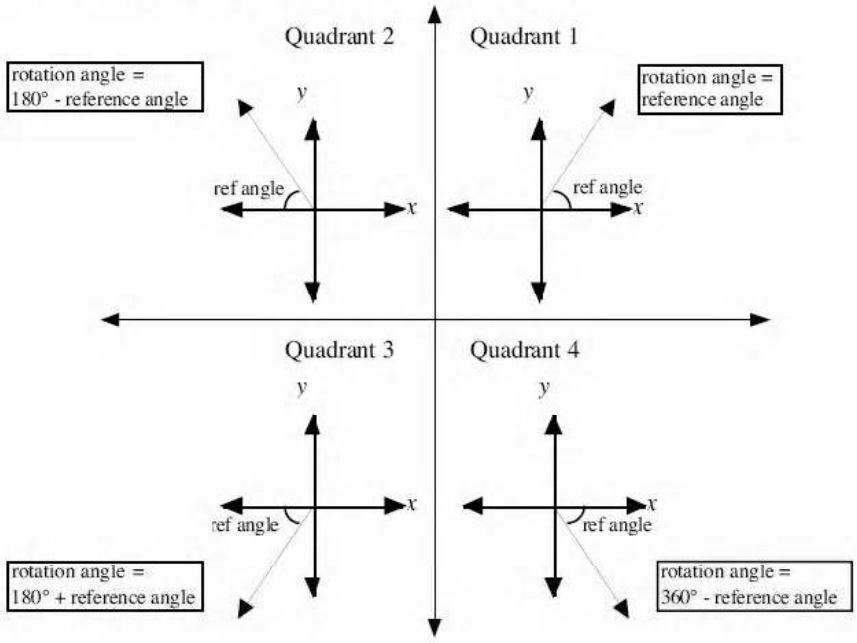
Trigonometry - Functions and Graphs Lesson #3:
Applications of Reference Angles and the CAST Rule

Warm-Up *Review of Reference Angles and the Cast Rule*

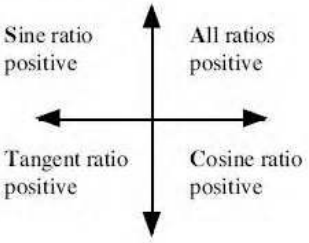
The reference angle for any rotation angle is the acute angle between the terminal arm of the rotation angle and the *x*-axis.



- **Memorize the relationship between reference angle and rotation angle in each quadrant.**
- **The relationship given between reference angle and rotation angle applies when the rotation angle is converted (where necessary) to a principal angle.**

The CAST rule may be used to determine the sign of a trigonometric ratio.

- the **CAST** rule or
- by remembering to "**A**dd **S**ugar **T**o **C**offee"



The reciprocal trigonometric ratios follow the same framework as their corresponding primary ratio.

Solving Equations Involving Sine or Cosine

We can use the concepts of reference angles and signs of the trigonometric ratio to solve equations involving sine or cosine. The following procedure may be used to solve an equation such as $\sin x = 0.5$, where $0° \le x \le 360°$.

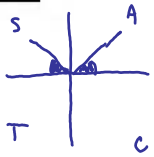
Step 1: Determine the quadrant(s) the angle will be in by looking at the sign of the ratio.

Step 2: Determine the reference angle (always between 0° and 90°) and draw a rough sketch in the appropriate quadrant(s). The reference angle is found as follows:
Use [2nd] [sin] or [2nd] [cos] of the **absolute value** of the given quantity.

Step 3: Determine the rotation angle(s) using the reference angle and the quadrant(s).

- Always check the given domain to determine which quadrants are valid in the calculation. As an example, sometimes the domain is restricted to $0° \le x \le 180°$, or $90° \le x \le 180°$.
- Pressing [sin] or [cos] then the angle [Enter] gives the value of the trigonometric ratio for that angle, whereas pressing [2nd] [sin] or [2nd] [cos] then the given value [Enter] determines the reference angle.

Use the above procedure to solve $\sin x = 0.5$, where $0° \le x \le 360°$



ref $x = 30°$

$x_1 = 0 + 30° = 30°$

$x_2 = 180 - 30° = 150°$

Find the measure of θ, to the nearest whole number, where $0° \le \theta \le 360°$.

a) $\sin \theta = -0.8090$

ref L = 54°

$\theta_1 = 180 + 54 = 234°$

$\theta_2 = 360 - 54 = 306°$

b) $\cos \theta = -0.8090$

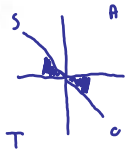
ref L = 36°

$\theta_1 = 180 - 36 = 144°$

$\theta_2 = 180 + 36 = 216°$

c) $\tan \theta = -2.4586$

ref L = 68°



$\theta_1 = 180 - 68° = 112°$

$\theta_2 = 360° - 68 = 292°$

Find the measure of A, to the nearest whole number, where $0° \le A \le 360°$.

a) $\cot A = 0.5$ $\quad \tan A = \frac{1}{.5} = 2$

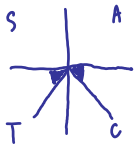
$A_1 = 0 + 63° = 63°$ $\quad$ ref $A = 63°$

$A_2 = 180 + 63° = 243°$

b) $\csc A = -2.86$ $\quad \sin A = -\frac{1}{2.86} = -0.3497$

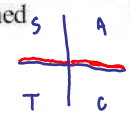
$A_1 = 180 + 20 = 200°$ $\quad$ ref $A = 20°$

$A_2 = 360 - 20 = 340°$



Find the measure of θ, to the nearest whole number, where $0° \le \theta \le 360°$.

a) $\csc \theta$ is undefined



$\sin\theta = 0$

IA we treat θ as positive quad 1/2

ref $\angle = 0°$

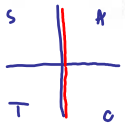
IA we treat θ as negative quad 3/4

ref $\angle = 0°$

both scenarios lead to the same Answer

$\theta = 0°, 180°, 360°$

b) $\cos \theta = 0$



ref $\theta = 90°$

$\theta_1 = 90°$

$\theta_2 = 270°$

c) $\sqrt{\sin^2\theta} = \sqrt{\frac{1}{2}}$

$\sin\theta = \pm\frac{1}{\sqrt{2}}$

ref $\theta = 45°$

$\theta_1 = 0 + 45° = 45°$

$\theta_2 = 180 - 45° = 135°$

$\theta_3 = 180 + 45° = 225°$

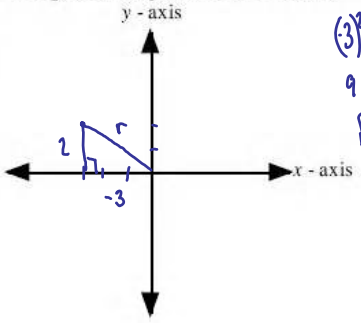
$\theta_4 = 360 - 45° = 315°$

Complete Assignment Questions #1 - #5

Finding the Exact Values of Trigonometric Ratios

Class Ex. #5

Given that the point $(-3, 2)$ lies on the terminal arm of θ, find the values of the primary and reciprocal trigonometric ratios.



$(3)^2+(2)^2=r^2$

$9+4=r^2$

$\sqrt{13}=\sqrt{r^2}$

$r=\sqrt{13}$

$\sin\theta=\frac{o}{h}=\frac{2}{\sqrt{13}}\cdot\frac{\sqrt{13}}{\sqrt{13}}=\frac{2\sqrt{13}}{13}$ $\csc\theta=\frac{h}{o}=\frac{\sqrt{13}}{2}$

$\cos\theta=\frac{a}{h}=\frac{-3}{\sqrt{13}}\cdot\frac{\sqrt{13}}{\sqrt{13}}=\frac{-3\sqrt{13}}{13}$ $\sec\theta=\frac{h}{a}=\frac{\sqrt{13}}{-3}$

$\tan\theta=\frac{o}{a}=\frac{2}{-3}$ $\cot\theta=\frac{a}{o}=\frac{-3}{2}$

Class Ex. #6

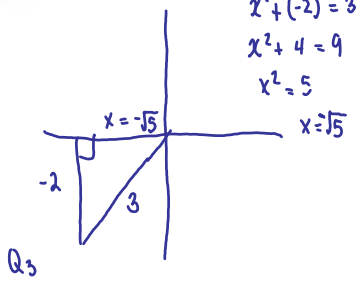
Angle A terminates in the third quadrant with $\sin A=-\frac{2}{3}$. Sketch a diagram and find the other five trigonometric ratios for angle A.

$x^2+(-2)^2=3^2$

$x^2+4=9$

$x^2=5$

$x=\sqrt{5}$



Q3

$\sin A=-\frac{2}{3}=\frac{o}{h}$ $\csc A=\frac{h}{o}=\frac{-3}{2}$

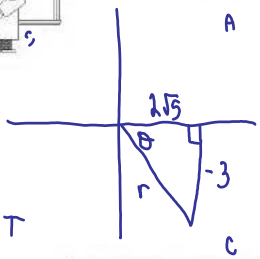
$\cos A=\frac{a}{h}=\frac{-\sqrt{5}}{3}$ $\sec A=\frac{h}{a}=\frac{-3}{\sqrt{5}}\cdot\frac{\sqrt{5}}{\sqrt{5}}=\frac{-3\sqrt{5}}{5}$

$\tan A=\frac{o}{a}=\frac{-2}{-\sqrt{5}}$ $\cot A=\frac{a}{o}=\frac{-\sqrt{5}}{-2}=\frac{\sqrt{5}}{2}$

$=\frac{2}{\sqrt{5}}\cdot\frac{\sqrt{5}}{\sqrt{5}}=\frac{2\sqrt{5}}{5}$

Class Ex. #7

If $\tan\theta=-\frac{3}{2\sqrt{5}}$ and $\sec\theta$ is positive, then find the value of $\sin\theta$.



A

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S

$(2\sqrt{5})^2+(-3)^2=r^2$

$20+9=r^2$

$29=r^2$

$r=\sqrt{29}$

$\sin\theta=\frac{o}{h}=\frac{-3}{\sqrt{29}}\times\frac{\sqrt{29}}{\sqrt{29}}=\frac{-3\sqrt{29}}{29}$

$2\sqrt{5}\times2\sqrt{5}$

$=4\sqrt{5}$

$=4\cdot5$

$=20$

Complete Assignment Questions #6 - #11

Assignment

1. Find the measure of θ, to the nearest degree, where $0° \le \theta \le 360°$.

a) $\sin \theta = 0.6485$ **b)** $\cos \theta = -0.8219$

c) $\tan \theta = 0.4668$ **d)** $\csc \theta = 1.0138$

2. Find the measure of A, to the nearest degree, where $0° \le A \le 360°$.

a) $\sec A = 1.2364$ **b)** $\cot A = -0.4458$

c) $\sin A = -1$ **d)** $\cot A$ is undefined

3. Find the measure of θ, to the nearest degree, where $0° \le \theta \le 360°$.

a) $\tan^2 \theta = 3$ **b)** $\sec^2 \theta = \frac{4}{3}$

4. Find the value of the following trigonometric ratios to 4 decimal places.

a) $\tan 30°$ **b)** $\sin(-180°)$ **c)** $\cot 78°$ **d)** $\sec 271°$ **e)** $\csc(-205°)$

5. Find the value of x. Note that x represents either an angle, where $0° \le x \le 360°$, or the value of a trigonometric ratio.

a) $\csc 150° = x$ **b)** $\cot x = -0.5638$ **c)** $x = \sec 348°$

d) $1.0736 = \tan x$ **e)** $\sin(-184°) = x$ **f)** $\sin x = 0$

6. Solve for the required ratios in each of the following. Express each answer as an exact value with a rational denominator.

a) θ is a second quadrant angle. If $\tan\theta = -\frac{\sqrt{3}}{5}$, find $\cos\theta$, $\csc\theta$, and $\cot\theta$.

b) θ is a fourth quadrant angle. If $\tan\theta = -\frac{\sqrt{3}}{5}$, find $\cos\theta$, $\csc\theta$, and $\cot\theta$.

c) If $\cos A = -\frac{7}{25}$, and $180° \le A \le 270°$, find the values of $\sin A$, $\tan A$, and $\sec A$.

7. If $\sin X = -\frac{1}{4}$ and $\cot X$ is positive, express $\cos X$ as an exact value.

Multiple Choice

8. The solution of the equation $\sec x = -10.366$ in the interval $0° \le x \le 360°$ is

A. $84°, 276°$
B. $96°, 264°$
C. $96°, 276°$
D. $264°, 276°$

9. If $\csc \theta = -2$, $270° \le \theta \le 360°$ the value of $\cos \theta$ is

A. $-\frac{1}{2}$
B. $-\frac{\sqrt{3}}{2}$
C. $\frac{1}{2}$
D. $\frac{\sqrt{3}}{2}$

10. Angle P has a terminal arm in the third quadrant.
If $\cot P = \sqrt{3}$, the value of $\sin P - \cos P$ is:

A. $\frac{1 - \sqrt{3}}{2}$
B. $\frac{\sqrt{3} - 1}{2}$
C. $\frac{1 + \sqrt{3}}{2}$
D. $\frac{-1 - \sqrt{3}}{2}$

Numerical Response

11. For $360° < \theta < 540°$, the solution, to the nearest degree, of the equation $\cos \theta = -\frac{1}{3}$ is ____ .

Answer Key

1. **a)** $40°, 140°$ **b)** $145°, 215°$
c) $25°, 205°$ **d)** $81°, 99°$

2. **a)** $36°, 324°$ **b)** $114°, 294°$
c) $270°$ **d)** $0°, 180°, 360°$

3. **a)** $60°, 120°, 240°, 300°$
b) $30°, 150°, 210°, 330°$

4. **a)** 0.5774 **b)** 0.0000 **c)** 0.2126 **d)** 57.2987 **e)** 2.3662

5. **a)** 2 **b)** $119°, 299°$ **c)** 1.0223 **d)** $47°, 227°$ **e)** 0.0698 **f)** $0°, 180°, 360°$

6. **a)** $\cos\theta = -\frac{5\sqrt{7}}{14}, \csc\theta = \frac{2\sqrt{21}}{3}, \cot\theta = -\frac{5\sqrt{3}}{3}$

b) $\cos\theta = \frac{5\sqrt{7}}{14}, \csc\theta = -\frac{2\sqrt{21}}{3}, \cot\theta = -\frac{5\sqrt{3}}{3}$

c) $\sin A = -\frac{24}{25}, \tan A = \frac{24}{7}, \sec A = -\frac{25}{7}$

7. $-\frac{\sqrt{15}}{4}$ **8.** B **9.** D **10.** B **11.** 469