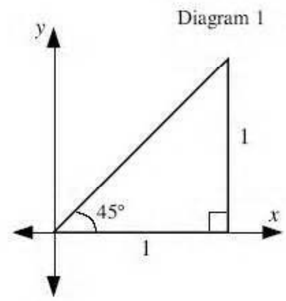
Trigonometry - Functions and Graphs Lesson 6:
Special Triangles, Exact Values, and the Unit Circle

Introduction

Our objective is to determine exact values for the trigonometric ratios of angles whose measure is a multiple of 30° $\left(\frac{\pi}{6}\text{ rad}\right)$ or a multiple of 45° $\left(\frac{\pi}{4}\text{ rad}\right)$.

Warm-up

a) Diagram 1 shows an angle of 45° in standard position. An isosceles triangle is drawn whose equal sides are 1 unit.



i) Determine the length of the hypotenuse.

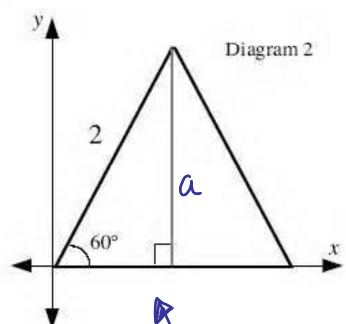
$r^2 = 1^2 + 1^2 = 2 \quad r = \sqrt{2} \quad \text{hypotenuse} = \sqrt{2}$

ii) Use SOHCAHTOA or the x, y, r formulas to complete:

sin 45° = $\frac{y}{r}$ cos 45° = $\frac{x}{y}$ tan 45° = $\frac{y}{x}$

$\frac{1}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$ $\quad \frac{1}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$ $\quad = \frac{1}{1} = 1$

b) Diagram 2 shows an angle of 60° in standard position. An equilateral triangle is drawn whose equal sides are 2 units and a vertical altitude is drawn which divides the equilateral triangle into two congruent triangles.



i) Determine the length of the altitude.

$a^2 = 2^2 - 1^2 = 3 \quad a = \sqrt{3} \quad \text{altitude} = \sqrt{3}$

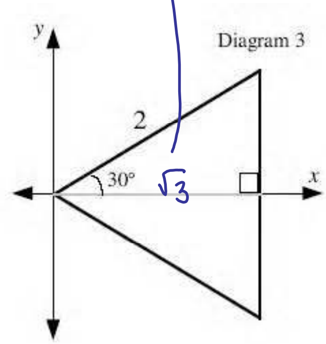
ii) Complete :

sin 60° = $\frac{y}{r}$ cos 60° = $\frac{x}{r}$ tan 60° = $\frac{y}{x}$

$= \frac{\sqrt{3}}{2}$ $\quad = \frac{1}{2}$ $\quad = \frac{\sqrt{3}}{1} = \sqrt{3}$

same

c) Diagram 3 shows an angle of 30° in standard position. An equilateral triangles is drawn whose equal sides are 2 units and a horizontal altitude is drawn which divides the equilateral triangle into two congruent triangles.



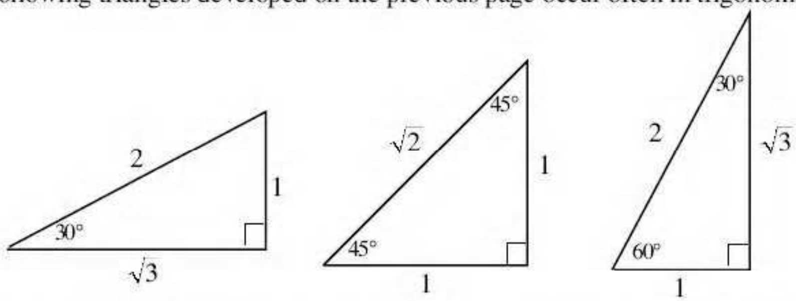
i) Complete:

sin 30° = $\frac{y}{r}$ cos 30° = $\frac{x}{r}$ tan 30° = $\frac{y}{x}$

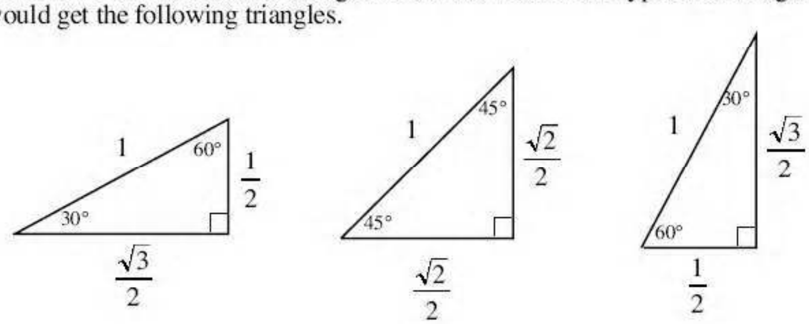
$= \frac{1}{2}$ $\quad = \frac{\sqrt{3}}{2}$ $\quad = \frac{1 \cdot \sqrt{3}}{\sqrt{3} \cdot \sqrt{3}} = \frac{\sqrt{3}}{3}$

Special Triangles

The following triangles developed on the previous page occur often in trigonometry



However if we consider similar triangles to the above, all with hypotenuse length of one unit, we would get the following triangles.



The triangles above are similar to the ones in the Warm-Up and illustrate the trigonometric ratios as exact values for angles of 30°, 45° and 60°.

In each diagram the horizontal distance is x, the vertical distance is y and the hypotenuse is $r = 1$.

Finding Exact Primary Trigonometric Values for 0° and 90°

a) Consider a rotation angle of 0°. In this case $x = 1$, $y = 0$ and $r = 1$.

sin 0°	$y/r = 0/1 = 0$
cos 0°	$x/r = \frac{1}{1} = 1$
tan 0°	$y/x = \frac{0}{1} = 0$

b) Consider a rotation angle of 90°. In this case $x = 0$, $y = 1$ and $r = 1$

sin 90°	$\frac{y}{r} = \frac{1}{1} = 1$
cos 90°	$x/r = \frac{0}{1} = 0$
tan 90°	$y/x = \frac{1}{0}$ undefined

c) Explain why tan 90° is undefined.

because $\frac{y}{x} = \frac{1}{0}$ undefined for division by zero

Memorizing The Angles and Ratios of Special Triangles

Although there are many ways to memorize the angles and ratios of special triangles, we will discuss two

- by chart or • by the unit circle

Using a Chart for Special Triangles

We can summarize the exact values of trig ratios of 0° (0 rad), 30° $\left(\frac{\pi}{6}\text{ rad}\right)$, 45° $\left(\frac{\pi}{4}\text{ rad}\right)$, 60° $\left(\frac{\pi}{3}\text{ rad}\right)$ and 90° $\left(\frac{\pi}{2}\text{ rad}\right)$ in the following chart.

x in degrees	0°	30°	45°	60°	90°
x in radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin x$	0	$\frac{1}{2}$	$\sqrt{2}/2$	$\sqrt{3}/2$	1
$\cos x$	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$\frac{1}{2}$	0
$\tan x$	0	$\sqrt{3}/3$	1	$\sqrt{3}/1$	undefined

This chart should be memorized.

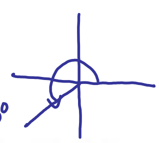
Determining Exact Values of Trigonometric Ratios Using the Chart

We can use the previous table together with the concept of reference angles and the CAST rule to determine the exact values of the trigonometric ratios of certain angles in quadrants 2, 3, and 4.


Class Ex. #1

Complete the solution below to find the exact value of :

a) sin 210°



ref∠ = 210 - 180 = 30°

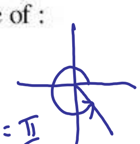
Solution

A rotation angle of 210° has a reference angle of 30°.

In quadrant 3 the sine ratio is negative.

sin 210° = – sin 30° = $-\frac{1}{2}$

b) $\cos \frac{5\pi}{3}$



ref θ = $2\pi - \frac{5\pi}{3} = \frac{\pi}{3}$

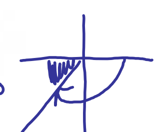
Solution

A rotation angle of $\frac{5\pi}{3}$ has a reference angle of $\pi/3$

In quadrant 4 the cosine ratio is positive.

$\cos\frac{5\pi}{3} = \cos\frac{\pi}{3} = 1/2$

c) tan (–135)°



180 - 135 = 45°

T positive

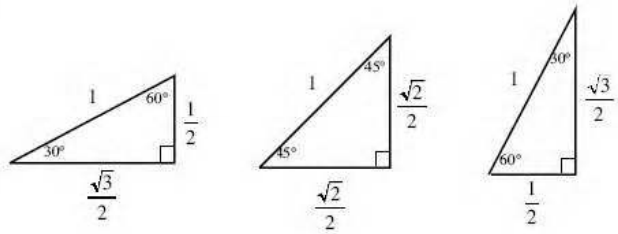
tan(-135) = tan 45° = 1

Complete Assignment Questions #1 - #3

Using the Unit Circle for Special Triangles

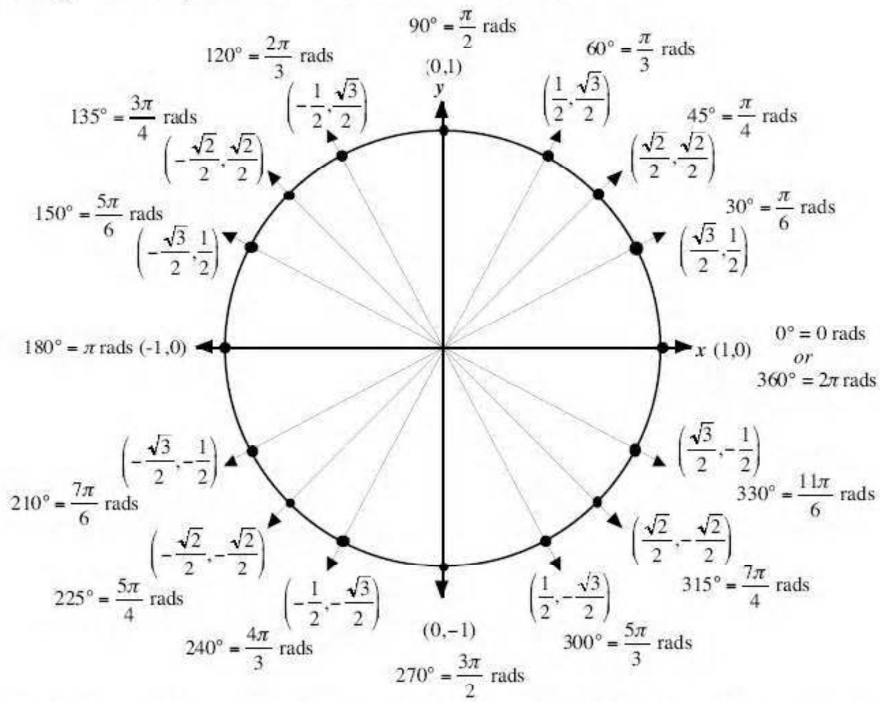
An alternative method for determining exact values for trigonometric ratios of angles greater than 90° is to use the **unit circle**.

Consider a rotating arm of length 1 unit on the triangles developed from the Warm-Up



In each diagram the;
- horizontal distance is x,
- vertical distance is y
- hypotenuse is $r = 1$.

These triangles can be placed in a circle with a radius of one unit.



The circle above, with a radius of one unit, is called the **unit circle** and is important you understand how it works.

Recall the formulas $\sin\theta = \frac{y}{r}$, $\cos\theta = \frac{x}{r}$, $\tan\theta = \frac{y}{x}$, and $\cot\theta = \frac{x}{y}$

Note

- In the unit circle, where $r = 1$, we have:

  $\sin\theta =$ $y/1 = y$ and $\cos\theta =$ $x/1 = x$

- Every point on the unit circle has coordinates (x, y) which can be written as $(\cos\theta, \sin\theta)$
- $\tan\theta = \frac{\sin\theta}{\cos\theta}$
- $\cot\theta = \frac{\cos\theta}{\sin\theta}$

Use the unit circle to find the exact value of all the trigonometric ratios for a rotation angle of 240°. Give each answer with a rational denominator.

sin 240° = $-\frac{\sqrt{3}}{2}$ $-\sin 60°$

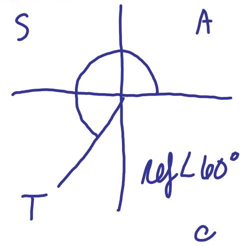
cos 240° = $-\frac{1}{2}$ $-\cos 60°$

tan 240° = $\frac{-\sqrt{3}}{-1} = \sqrt{3}$ $\tan 60$

csc 240° = $-\frac{2}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$ $-\csc 60°$

sec 240° = $-\frac{2}{1}$ $-\sec 60°$

cot 240° = $\frac{1}{\sqrt{3} \times \sqrt{3}} \sqrt{3} = \frac{\sqrt{3}}{3}$ $\cot 60°$

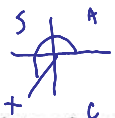


Use the unit circle to find the exact value of

a) $\cos \frac{3\pi}{4}$ $-\cos\frac{\pi}{4}$

$-\frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{-\sqrt{2}}{2}$

b) tan 600°   tan 240   tan 60° $\frac{\sqrt{3}}{1}$



c) csc 3π $= \frac{1}{\sin\pi} = \frac{1}{0}$ undefined

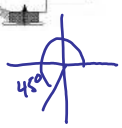
We now have two methods for determining exact values of trigonometric ratios of angles greater than 90°. Use either method.

Use the chart or unit circle to find the exact value of :

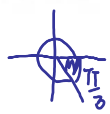
a) sec 225° $= \frac{1}{\cos 225} = -\frac{1}{\cos 45} = -\frac{1}{\frac{1}{\sqrt{2}}} = -\sqrt{2}$

cos   sin

$\frac{1}{\sqrt{2}} \frac{x}{r}$ $\frac{1}{\sqrt{2}} \frac{y}{r}$



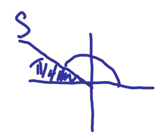
b) $\cot \frac{5\pi}{3}$ $= -\frac{1}{\tan\frac{\pi}{3}} = -\frac{1}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{-\sqrt{3}}{3}$



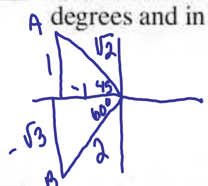
c) $\sin^2 \frac{3\pi}{4} + \cos^2 \frac{3\pi}{4}$

$\left(+\sin\frac{\pi}{4}\right)^2 + \left(-\cos\frac{\pi}{4}\right)^2$

$\left(\frac{1}{\sqrt{2}}\right)^2 + \left(-\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2} + \frac{1}{2} = \frac{2}{2} = 1$



$A\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ and $B\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$ are two points on the unit circle. If an object rotates counterclockwise from point A to point B through what angle has it rotated? Answer in degrees and in radians.



180 − 45 = 135

180 + 60 = 240

angle = 240° − 135° = 105°

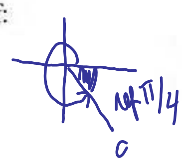
$105 \times \frac{\pi}{180} = \frac{7\pi}{12}$

Find the exact value of:

a) $\log_2\left(\cos\frac{7\pi}{4}\right)$

$\log_2\left(\cos\frac{\pi}{4}\right)$

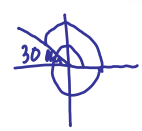
$\log_2 \frac{1}{\sqrt{2}} = \log_2 2^{-\frac{1}{2}} = -\frac{1}{2}$



b) $\log_4(\csc 510°)$

$\log_4 \csc 30°$

$\log_4 2 = \log_4 4^{\frac{1}{2}} = \frac{1}{2}$



$510 - 360 = 150$

Using Exact Values to Solve Simple Trigonometric Equations

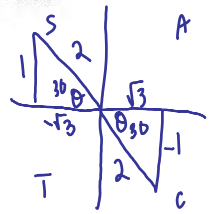
Find the exact value(s) of θ where

a) $\cot\theta = -\sqrt{3}$, $0° \le \theta \le 360°$

$\frac{a}{o}$ $\quad \frac{\sqrt{3}}{1}$

$\theta_1 = 180 - 30 = 150°$

$\theta_2 = 360 - 30 = 330°$

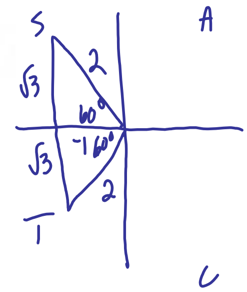


b) $\sec\theta = -2$, $0 \le \theta \le 2\pi$.

$\cos\theta = \frac{-1}{2} \frac{a}{h}$

$\theta_1 = 180 - 60 = 120°$

$\theta_2 = 180 + 60 = 240°$



$120 \times \frac{\pi}{180} = \frac{2\pi}{3}$ $\quad 240 \times \frac{\pi}{180} = \frac{4\pi}{3}$

Complete Assignment Questions #4 - #15

Assignment

1. Find the exact value of :

a) cos 120° **b)** tan 300° **c)** sin 135°

d) sin (- 30)° **e)** $\cos^2 225°$ **f)** tan 480°

2. Find the exact value of :

a) $\sin\frac{5\pi}{3}$ **b)** $\tan\frac{7\pi}{6}$ **c)** $\cos\left(-\frac{2\pi}{3}\right)$

d) $\sin\left(-\frac{\pi}{6}\right)$ **e)** $\tan^2\frac{2\pi}{3}$ **f)** $\cos\left(-\frac{5\pi}{3}\right)$

6

3. Find the exact value of :

a) $\sec 300°$ **b)** $\cot \frac{5\pi}{6}$ **c)** $\csc\left(-\frac{5\pi}{3}\right)$

d) $\cot 930°$ **e)** $\sec \frac{3\pi}{2}$ **f)** $\csc 5\pi$

4. Find exact values for each using the unit circle.

a) $\cos 60°$ **b)** $\sin 90°$ **c)** $\csc \frac{\pi}{6}$ **d)** $\cot \frac{7\pi}{4}$

e) $\sec 150°$ **f)** $\tan \frac{5\pi}{3}$ **g)** $\csc(-330°)$ **h)** $\sec(-135°)$

i) $\cot 540°$ **j)** $\tan\left(-\frac{\pi}{3}\right)$

5. Find the coordinates of the point on the unit circle that correspond to each rotation. Give the coordinates correct to two decimal places where necessary.

a) $-270°$ **b)** $103°$ **c)** $-16°$

6. The point $T(-0.8829, 0.4695)$ lies on the unit circle. Determine the value of θ where θ is the angle made by the positive x-axis and the line passing through T.

7. $P\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$ and $Q\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ are two points on the unit circle. If an object rotates counterclockwise from point P to point Q through what angle has it rotated? Answer in degrees and in radians.

8. Determine the length of the arc on the unit circle which has initial point $A\,(-1, 0)$ and terminal point $B\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$.

9. Find the exact value of:

a) $\log_3\left(\cot\frac{4\pi}{3}\right)$

b) $\sum_{n=4}^{8} \sin\frac{n\pi}{6}$

10. Find the exact value(s) of θ where $0° \le \theta \le 360°$.

a) $\sin\theta = 1$

b) $\cos\theta = -1$

c) $\cot^2\theta = 1$

e) $\sec\theta = \frac{2}{\sqrt{2}}$

11. Find the measure of θ where $0 \le \theta \le 2\pi$.

a) $\sin\theta = -\frac{\sqrt{3}}{2}$

b) $\tan\theta = 0$

c) $\cos\theta = -\frac{1}{2}$

d) $\csc\theta = \sqrt{2}$

e) $\cot\theta = -1$

f) $\cot\theta$ is undefined

12. If $\cos\theta = \dfrac{1}{\sqrt{2}}$, and θ is not a first quadrant angle, find the exact values of $\cot\theta$ and $\csc\theta$.

Multiple Choice

13. Correct to the nearest tenth of a radian, the smallest positive root of the equation $\tan x + \sqrt{3} = 0$ is

A. $\dfrac{\pi}{3}$

B. $\dfrac{2\pi}{3}$

C. $\dfrac{4\pi}{3}$

D. $-\dfrac{\pi}{3}$

14. If the point $A\left(\dfrac{\pi}{6}, -2\right)$ lies on the graph of $f(x) = \sin(x + \pi) - d$, then the value of d is

A. $\dfrac{5}{2}$

B. $\dfrac{3}{2}$

C. $\dfrac{2-\sqrt{3}}{2}$

D. $\dfrac{2-\sqrt{2}}{2}$

Numerical Response

15. The solution to the equation $\cot\theta = -\sqrt{3}$, $180° < \theta < 360°$, to the nearest degree, is _____ .

Answer Key

1. a) $-\frac{1}{2}$ b) $-\sqrt{3}$ c) $\frac{\sqrt{2}}{2}$ d) $-\frac{1}{2}$ e) $\frac{1}{2}$ f) $-\sqrt{3}$

2. a) $-\frac{\sqrt{3}}{2}$ b) $\frac{\sqrt{3}}{3}$ c) $-\frac{1}{2}$ d) $-\frac{1}{2}$ e) 3 f) $\frac{1}{2}$

3. a) 2 b) $-\sqrt{3}$ c) $\frac{2\sqrt{3}}{3}$ d) $\sqrt{3}$ e) undefined f) undefined

4. a) $\frac{1}{2}$ b) 1 c) 2 d) -1 e) $-\frac{2\sqrt{3}}{3}$
f) $-\sqrt{3}$ g) 2 h) $-\sqrt{2}$ i) undefined j) $-\sqrt{3}$

5. a) (0,1) b) (–0.22, 0.97) c) (0.96, –0.28)

6. 152° **7.** 210° or $\frac{7\pi}{6}$ **8.** $\frac{2\pi}{3}$ units

9. a) $-\frac{1}{2}$ b) 0

10. a) 90° b) 180° c) 45°, 135°, 225°, 315° d) 45°, 315°

11. a) $\frac{4\pi}{3}, \frac{5\pi}{3}$ b) $0, \pi, 2\pi$ c) $\frac{2\pi}{3}, \frac{4\pi}{3}$ d) $\frac{\pi}{4}, \frac{3\pi}{4}$ e) $\frac{3\pi}{4}, \frac{7\pi}{4}$ f) $0, \pi, 2\pi$

12. $\cot\theta = -1$ $\csc\theta = -\sqrt{2}$ **13.** B **14.** B **15.** 330