Trigonometry - Functions and Graphs Lesson #9:
Transformations of Trigonometric Functions Part 2

Warm-Up ***Introduction***

In this lesson we will consider the graphs of the functions whose equations are

$$y = a\sin[b(x-c)] + d \qquad \text{and} \qquad y = a\cos[b(x-c)] + d$$

and relate them to the graphs of the functions whose equations are $y = \sin x$ and $y = \cos x$.

In the first part of the lesson we concentrate on the effects of the parameters c and d.

a) Describe how the graph of the given function compares to the graph of $y = \sin x$, where x is in degrees

i) $y = \sin(x - 30°)$ hor trans 30° right

ii) $y = \sin x + 2$ Vert trans 2 up

iii) $y = \sin(x + 60°) - 1$ hor trans 60° left, Vert trans 1 down

iv) $y - 45 = \sin(x - 45°)$ hor trans 45° right, Vert tran 45 up

$y = \sin(x - 45°) + 45$

In trigonometry;
- a horizontal translation is called a **horizontal phase shift**, and,
- a vertical translation is called a **vertical displacement**.

Complete the table to describe how the graph of the given function compares to the graph of $y = \sin x$ where x is in radians. Use a graphing calculator if necessary.

Equation	Horizontal Phase Shift	Vertical Displacement
$y = \sin x$	0	0
$y = \sin\left(x - \frac{\pi}{4}\right)$	$\frac{\pi}{4}$ radians right	0
$y = \sin x + 5$	0	5 units up
$y + \cancel{\pi} = \sin\left(x + \frac{3\pi}{2}\right) - \pi$	$\frac{3\pi}{2}$ radians left	π units down
$y = \sin(x - c) + d$	c radians right	d units up
$y = a\sin[b(x - c)] + d$	c radians right	d units up

Would you expect similar effects on the graph of $y = a\cos[b(x - c)] + d$? Investigate if necessary.

Effects of c and d in $y = a\sin[b(x-c)] + d$ and $y = a\cos[b(x-c)] + d$

Changing the parameter "*c*" on the graphs of

$y = a\sin[b(x-c)] + d$ **and** $y = a\cos[b(x-c)] + d$

results in a horizontal phase shift with the following:

- a horizontal phase shift to the right if $c > 0$
- a horizontal phase shift to the left if $c < 0$

Changing the parameter "*d*" on the graphs of

$y = a\sin[b(x-c)] + d$ **and** $y = a\cos[b(x-c)] + d$

results in a vertical displacement with the following:

- a vertical displacement up if $d > 0$
- a vertical displacement down if $d < 0$

The vertical displacement can be determined from a graph using the formula $d = \dfrac{\text{Max} + \text{Min}}{2}$.

Summary of the Effects of the Parameters a, b, c, and d

For $y = a\sin[b(x-c)] + d$
$y = a\cos[b(x-c)] + d$

amplitude $= |a| = \dfrac{\text{Max} - \text{Min}}{2}$

period $= \dfrac{360°}{|b|}$ (for degree measure)

period $= \dfrac{2\pi}{|b|}$ (for radian measure)

horizontal phase shift $= c$

- to the right if $c > 0$
- to the left if $c < 0$

vertical displacement $= d$

- up if $d > 0$
- down if $d < 0$
- $d = \dfrac{\text{Max} + \text{Min}}{2}$

For $y = a\tan b[(x-c)] + d$

amplitude - not applicable

a value represents
- a vertical expansion or,
- a vertical compression

period $= \dfrac{180°}{|b|}$ (for degree measure)

period $= \dfrac{\pi}{|b|}$ (for radian measure)

horizontal phase shift $= c$

- to the right if $c > 0$
- to the left if $c < 0$

vertical displacement $= d$

- up if $d > 0$
- down if $d < 0$

Consider equations of the form $y = a\sin[b(x-c)] + d$ and $y = a\cos[b(x-c)] + d$, where $a = 1$, and $b = 1$. Write the equation which represents;

a) a cosine function having a horizontal phase shift of 75° right

$y = \cos(x - 75°)$

b) a sine function having a horizontal phase shift of $\frac{3\pi}{5}$ radians left, and a vertical displacement 4 units up

$y = \sin\left(x + \frac{3\pi}{5}\right) + 4$

Find the amplitude, period, horizontal phase shift, and vertical displacement of the graphs of the following functions defined on $x \in \Re$.

a) $y = 2\sin 3(x + \pi) - 4$

$a = 2$ amplitude 2 units

$b = 3$ period $\frac{2\pi}{|b|} = \frac{2\pi}{3}$ radians

$c = -\pi$ hor phase shift π radians left

$d = -4$ Vertical displacement 4 units down

b) $y = -\frac{2}{3}\cos\frac{1}{4}\left(x - \frac{\pi}{12}\right) + 3$

$a = -\frac{2}{3}$ amplitude $\frac{2}{3}$ units

$b = \frac{1}{4}$ period $\frac{2\pi}{|b|} = \frac{2\pi}{\frac{1}{4}} = 8\pi$

$c = \frac{\pi}{12}$ hor phase shift $\frac{\pi}{12}$ radians right

$d = 3$ Vertical displacement 3 units up

Find the amplitude, period, horizontal phase shift, and vertical displacement of the graphs of the following functions defined on $x \in \Re$.

a) $y = 2\sin(3x + \pi) - 4$

$y = 2\sin 3\left(x + \frac{\pi}{3}\right) - 4$

$a = 2$ amp 2 units

$b = 3$ period $\frac{2\pi}{3}$

$c = -\frac{\pi}{3}$ hor ph. shift $\frac{\pi}{3}$ to left

$d = -4$ vert Dis 4 units down

b) $y = -\cos\left(2x - \frac{\pi}{2}\right) + \pi$

$y = -\cos 2\left(x - \frac{\pi}{4}\right) + \pi$

$a = -1$ amp 1 unit

$b = 2$ period $\frac{2\pi}{2} = \pi$

$c = \frac{\pi}{4}$ hor ph shift $\frac{\pi}{4}$ rad right

$d = \pi$ vert Dis π units up

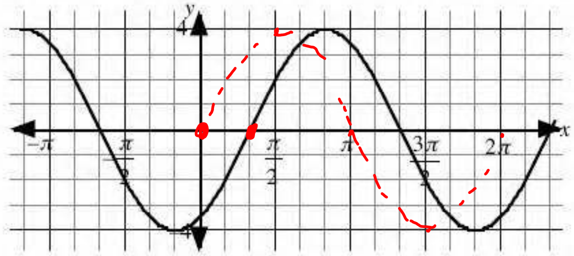
c) Compare the answer to Class Ex. #4a and Class Ex. #5a.

Complete Assignment Questions #1 - #2

Class Ex. #6

The graphs from a) - d) represent the same trigonometric function.

a) Write the equation of the graph in the form $y = a \sin(x - c)$ if $a > 0$ and there is a minimum possible horizontal phase shift.

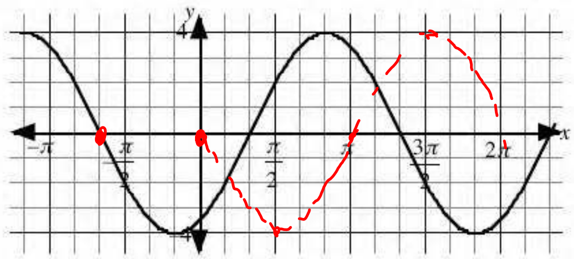


amp 4 $\quad a=4$

hor ph. shift $\frac{\pi}{3}$ right $\quad c=\frac{\pi}{3}$

$y = 4\sin\left(x - \frac{\pi}{3}\right)$

b) Write the equation of the graph in the form $y = a \sin(x - c)$ if $a < 0$ and there is a minimum possible horizontal phase shift.

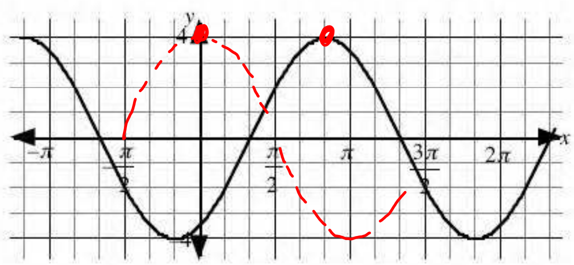


amp 4 $\quad a=-4$

hor ph. shift $\frac{2\pi}{3}$ left $\quad c=-\frac{2\pi}{3}$

$y = -4\sin\left(x + \frac{2\pi}{3}\right)$

c) Write the equation of the graph in the form $y = a \cos(x - c)$ if $a > 0$ and there is a minimum possible horizontal phase shift.

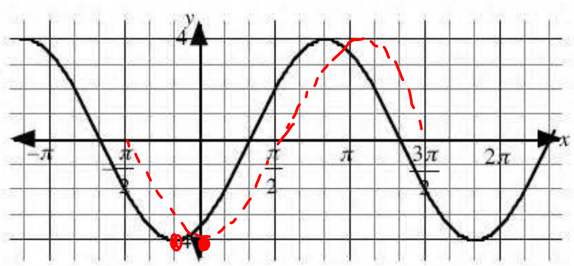


amp $=4 \quad a=4$

hor ph shift $\frac{5\pi}{6}$ right $\quad c=\frac{5\pi}{6}$

$y = 4\cos\left(x - \frac{5\pi}{6}\right)$

d) Write the equation of the graph in the form $y = a \cos(x - c)$ if $a < 0$ and there is a minimum possible horizontal phase shift.



amp $= 4 \quad a=-4$

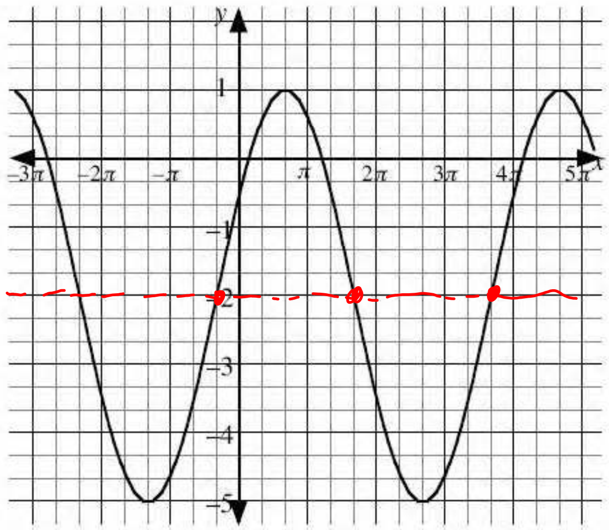
hor phase shift $\frac{\pi}{6}$ left $\quad c=-\frac{\pi}{6}$

$y = -4\cos\left(x + \frac{\pi}{6}\right)$

Class Ex. #7

Consider the graph shown.

a) If the graph represents a sine function where $a > 0$, write the equation represented by the graph.

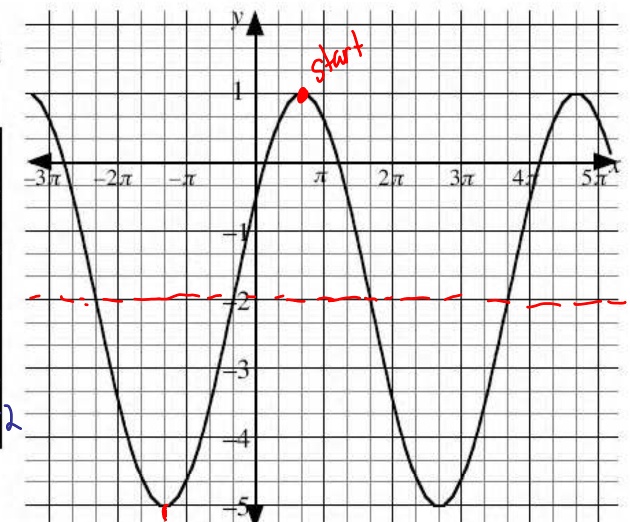


Amplitude	$\frac{1-(-5)}{2} = 3$
Period	$12\left(\frac{\pi}{3}\right) = 4\pi$
Min Horizontal Phase Shift	$\frac{\pi}{3}$ rad left
Vertical Displacement	$\frac{1+(-5)}{2} = -2$
Equation	$y = 3\sin\left[\frac{1}{2}\left(x+\frac{\pi}{3}\right)\right]-2$

b) If the graph in a) represents a sine function where $a < 0$, write the equation represented by the graph.

phase shift $= \frac{5\pi}{3}$ rad right $y = -3\sin\left[\frac{1}{2}\left(x-\frac{5\pi}{3}\right)\right]-2$

c) If the graph represents a cosine function where $a > 0$, write the equation represented by the graph.



Amplitude	3
Period	4π
Min Horizontal Phase Shift	$\frac{2\pi}{3}$ rad right
Vertical Displacement	-2
Equation	$y = 3\cos\left[\frac{1}{2}\left(x-\frac{2\pi}{3}\right)\right]-2$

$b = \frac{2\pi}{\text{period}} = \frac{2\pi}{4\pi} = \frac{1}{2}$

p.s $a<0$ p.s. $a>0$

d) If the graph in c) represents a cosine function where $a < 0$, write the equation represented by the graph.

phase shift to $\frac{4\pi}{3}$ left $y = -3\cos\left(\frac{1}{2}\left(x+\frac{4\pi}{3}\right)\right)-2$

Class Ex. #8

Consider the graphs of the functions $y = a \sin [b(x - c)] + d$ and $y = a \cos [b(x - c)] + d$.

a) Changing which of the parameters a, b, c and d affect the:

i) domain **ii)** range **iii)** amplitude **iv)** period **v)** zeros

none a, d a b b, c, d a if $d \neq 0$

b) State the maximum and minimum values of the functions in terms of a, b, c, and d, if $a > 0$.

max $a(1) + d = a + d$

min $a(-1) + d = -a + d$

c) Determine the range of the function $y = 3 \sin 2(x - \pi) - 4$.

max $= 3(1) - 4 = -1$

min $= 3(-1) - 4 = -7$

range $\{y \mid -7 \leq y \leq -1, y \in \mathbb{R}\}$

Complete Assignment Questions #3 - #12

Assignment

1. Determine the amplitude, period, horizontal phase shift, and the vertical displacement for each function.

a) $y = \cos\left(x - \frac{\pi}{4}\right) + 3$ **b)** $y = 3 \cos \frac{1}{2}\left(x - \frac{\pi}{2}\right)$ **c)** $y = 3 \cos \frac{1}{2}x - \frac{\pi}{2}$

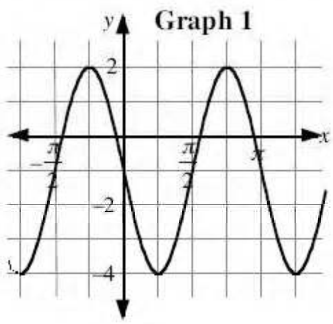
d) $y = \sin\left(4x - \frac{\pi}{2}\right)$ **e)** $y = -2 \cos 3(x - 45°) + 4$ **f)** $y = 7 \sin\left(\frac{1}{4}x + 20°\right) - 1$

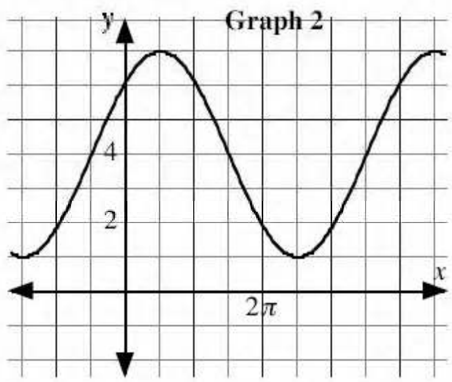
2. a) Find the equation of a sine function that has a vertical displacement 3 units up, a horizontal phase shift of 60° to the left, a period of 210° and an amplitude of 4.

b) Find the equation of a cosine function that has a vertical displacement 5 units down, a horizontal phase shift of $\frac{2\pi}{3}$ radians to the right, a period of $\frac{5\pi}{4}$ and an amplitude of 3.

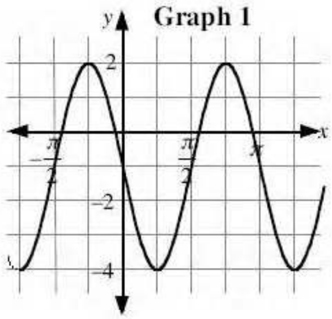
3. Graphs 1 and 2 each represent the graphs of trigonometric functions.

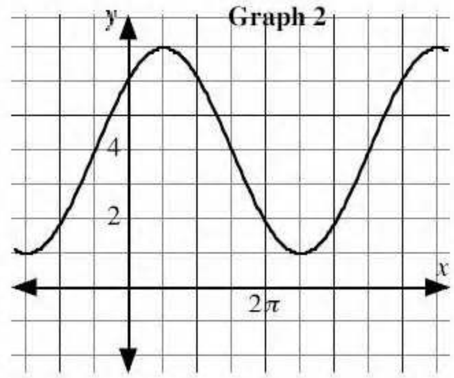
a) Assuming a minimum possible phase shift, write the equation of each graph in the form $y = a \sin[b(x - c)] + d$ if: **i)** $a > 0$ **ii)** $a < 0$



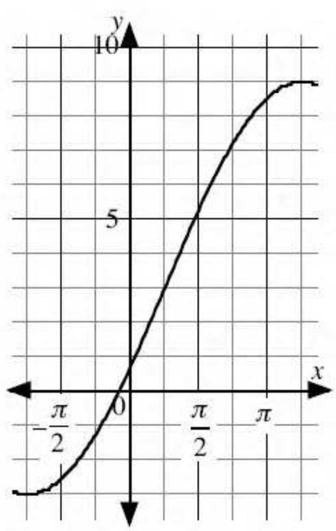


b) Assuming a minimum possible phase shift, write the equation of each graph in the form $y = a\cos[b(x-c)] + d$ if: **i)** $a > 0$ **ii)** $a < 0$

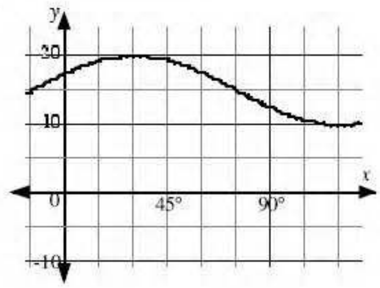




4. The cosine graph shown has a range $-3 \le y \le 9$.The graph has an equation in the form $y = a\cos[b(x-c)] + d, a > 0$. Determine the equation if the graph has a minimum possible phase shift.

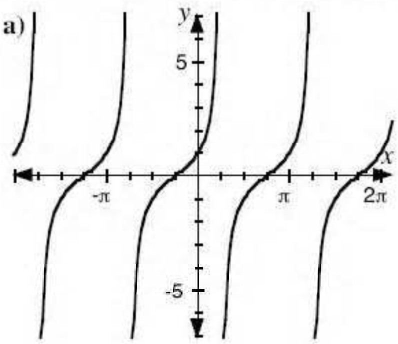


5. The sine graph shown has a maximum value of 20 and a minimum value of 10. The graph has an equation in the form $y = a \sin [b(x - c)] + d$ with $a > 0$. Determine the equation if the graph has a minimum possible phase shift.

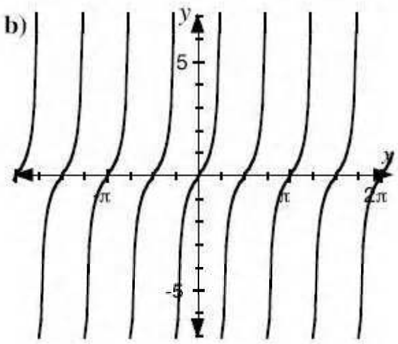


6. The graphs shown have an equation in the form $y = \tan b(x - c)$. Determine the equation of each graph.

a)



b)



7. Determine the range of the functions represented below.

a) $y = 2 \sin x - 2$

b) $y = 3 \cos \dfrac{1}{2}\left(x - \dfrac{\pi}{2}\right) + 1$

c) $y = -\dfrac{1}{2} \cos 4(x - \pi) - 3$

d) $y = a \sin [b(x - c)] + d$

Multiple Choice

8. Which of the following graphs has the same x-intercepts as the graph of $y = \cos x$?

A. $y = \cos 4x$

B. $y = 4 \cos x$

C. $y = \cos x + 4$

D. $y = \cos (x + 4)$

9. Which equation is a tangent function with period $\dfrac{\pi}{3}$, and a vertical displacement -3?

A. $y = \tan \dfrac{\pi}{3} x - 3$

B. $y = \tan 3(x - 3)$

C. $y = \tan 3x - 3$

D. $y = \tan 6x - 3$

10. The equation $y = \pi \cos (\pi x - \pi)$ has a period and a horizontal phase shift to the right, respectively, of

A. π and π
B. π and 1
C. 2 and π
D. 2 and 1

11. Which statement concerning the graph of $y = -4 \cos \frac{x}{2} + 2$ is <u>not</u> correct?

A. The maximum value is 6.
B. The period is 4π.
C. The amplitude is -4
D. The vertical displacement is 2.

Numerical Response **12.** The period, to the nearest tenth, of the function $y = \sin 0.25x$, where x is in radians, is ______ .

Answer Key

1.

	amplitude	period	phase shift	vertical displacement
a)	1	2π	$\frac{\pi}{4}$ right	3 up
b)	3	4π	$\frac{\pi}{2}$ right	0
c)	3	4π	0	$\frac{\pi}{2}$ down
d)	1	$\frac{\pi}{2}$	$\frac{\pi}{8}$ right	0
e)	2	120°	45° right	4 up
f)	7	1440°	80° left	1 down

2. **a)** $y = 4 \sin \frac{12}{7}(x + 60°) + 3$

b) $y = 3 \cos \frac{8}{5}\left(x - \frac{2\pi}{3}\right) - 5$

(Note for **a)** and **b)**: the value of a can also be negative)

3. **a)** **i)** for $a > 0$, **Graph 1** $y = 3 \sin 2\left(x \pm \frac{\pi}{2}\right) - 1$, **Graph 2** $y = 3 \sin \frac{1}{2}\left(x + \frac{\pi}{2}\right) + 4$

ii) for $a < 0$, **Graph 1** $y = -3 \sin 2x - 1$, **Graph 2** $y = -3 \sin \frac{1}{2}\left(x - \frac{3\pi}{2}\right) + 4$

b) **i)** for $a > 0$, **Graph 1** $y = 3 \cos 2\left(x + \frac{\pi}{4}\right) - 1$, **Graph 2** $y = 3 \cos \frac{1}{2}\left(x - \frac{\pi}{2}\right) + 4$

ii) for $a < 0$, **Graph 1** $y = -3 \cos 2\left(x - \frac{\pi}{4}\right) - 1$, **Graph 2** $y = -3 \cos \frac{1}{2}\left(x + \frac{3\pi}{2}\right) + 4$

4. $y = 6 \cos \frac{1}{2}\left(x - \frac{5\pi}{4}\right) + 3$ **5.** $y = 5 \sin 2(x + 15°) + 15$

6. **a)** $y = \tan\left(x + \frac{\pi}{4}\right)$ **b)** $y = \tan 2x$

7. **a)** $\{y \mid -4 \le y \le 0, y \in R\}$ **b)** $\{y \mid -2 \le y \le 4, y \in R\}$

c) $\left\{y \mid -\frac{7}{2} \le y \le -\frac{5}{2}, y \in R\right\}$ **d)** $\{y \mid -a + d \le y \le a + d, y \in R\}$

8. B **9.** C **10.** D **11.** C **12.** 25.1