Trigonometry - Equations, Identities, and Modelling Lesson #2: Solving Second Degree Trigonometric Equations

Warm-Up

Introduction

In this lesson we will be solving **second degree** equations where the power of the trigonometric function is two (eg. $\sin^2 x - 3\sin x = 0$). We will:

- **factor** trigonometric expressions algebraically,
- use a **graphical approach** to determine an approximate solution,
- use **factoring** to determine solutions on a domain of length 2π radians, and,
- find the **general solution** over the domain of real numbers.

Trigonometric equations which can be solved by using identities will be covered in lesson 6.

Factoring Trigonometric Expressions

Just as with polynomial expressions, trigonometric expressions can be factored. The ability to factor trigonometric expressions is a useful skill in two areas:

- solving trigonometric equations (in this lesson)
- proving complicated identities (in lesson 6)

In factoring trigonometric expressions we can apply three basic factoring techniques:

- common factor,
- difference of two squares, and
- factoring trinomials of the form $ax^2 + bx + c, a \neq 0$.

$x^2 - y^2$
$(x+y)(x-y)$

Factor the following trigonometric expressions:

a) $8\tan A + 4$

$4(2\tan A + 1)$

b) $\sin^2 x - 3\sin x$

$\sin x(\sin x - 3)$

c) $4\sin^2 x - 1$

$(2\sin x - 1)(2\sin x + 1)$

d) $\csc^2 x - 3\csc x - 28$

$-28 = -7 \times 4$
$-3 = -7 + 4$

$(\csc x - 7)(\csc x + 4)$

e) $2\cos^2 x + 7\cos x - 4$

$-8 \quad 8 \times -1$
$7 \quad 8 + -1$

$(2\cos^2 x + 8\cos x)(-1\cos x - 4)$

$2\cos x(\cos x + 4) - 1(\cos x + 4)$

$(\cos x + 4)(2\cos - 1)$

Solving a Second Degree Equation Using a Graphical Approach

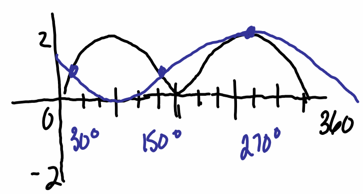
Consider the equation $2\sin^2 x = 1 - \sin x$.

$0 \le x \le 360$

a) Use a **graphical** approach to find the solution to the equation where $0 \le x \le 2\pi$. Give solutions as exact values.

$y_1 = 2(\sin(x))^2$

$y_2 = 1 - \sin(x)$



$x = 30 \times \frac{\pi}{180} = \frac{\pi}{6}$ $\quad x = 270 \times \frac{\pi}{180} = \frac{3\pi}{2}$

$x = 150 \times \frac{\pi}{180} = \frac{5\pi}{6}$

b) State the general solution to the equation.

$x = \frac{\pi}{6} + 2n\pi, \frac{5\pi}{6} + 2n\pi, \frac{3\pi}{2} + 2n\pi, n \in I$

Solving a Second Degree Equation Using an Algebraic Approach

Consider the equation $2\sin^2 x = 1 - \sin x$.

a) Use an **algebraic** approach to find the solution to the equation where $0 \le x \le 2\pi$. Give solutions as exact values.

$-2 = 2x - 1$

$1 = 2 + -1$

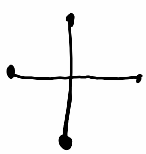
$2\sin^2 x + 1\sin x - 1 = 0$

$(2\sin^2 x + 2\sin x)(-1\sin x - 1)$

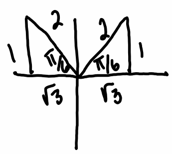
$2\sin x(\sin x + 1) - 1(\sin x + 1)$

$\sin x + 1 = 0 \quad 2\sin x - 1 = 0$

$\sin x = -1 \quad \sin x = \frac{1}{2} \frac{o}{h}$



$x = \frac{3\pi}{2}$



$x = \frac{\pi}{6}$

$x = \frac{6\pi}{6} - \frac{\pi}{6} = \frac{5\pi}{6}$

b) State the general solution to the equation.

$x = \frac{3\pi}{2} + 2n\pi, \frac{\pi}{6} + 2n\pi, \frac{5\pi}{6} + 2n\pi, n \in I$

Class Ex. #4

In each of the following:

a) Use an algebraic procedure to find the solution to the equation on the given domain.

b) Write the general solution to the equation

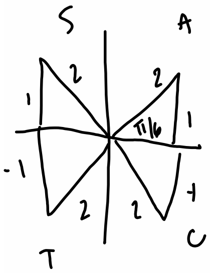
i) $4\sin^2 A - 1 = 0,\ 0 \le A \le 2\pi$

$(2\sin A - 1)_{=0}(2\sin A + 1)_{=0}$

$\sin A = \frac{1}{2}$ $\sin A = -\frac{1}{2}$ ref $\angle = \frac{\pi}{6}$

$A = \frac{\pi}{6}$ $A = \frac{6\pi}{6} + \frac{\pi}{6} = \frac{7\pi}{6}$

$A = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$ $A = \frac{12\pi}{6} - \frac{\pi}{6} = \frac{11\pi}{6}$



gen Soln $\frac{\pi}{6} + 2n\pi,\ \frac{5\pi}{6} + 2n\pi,\ \frac{7\pi}{6} + 2n\pi,\ \frac{11\pi}{6} + 2n\pi,\ n \in I$

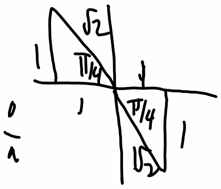
ii) $\tan^2 x + \tan x = 0,\ \ 0 \le x \le 2\pi$

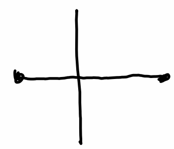
$\tan x(\tan x + 1)$

$\tan x = 0$ $\tan x = -\frac{1}{1}$

$x = 0$ $x = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$

$x = \pi$ $x = 2\pi - \frac{\pi}{4} = \frac{7\pi}{4}$





gen Soln $x = n\pi,\ \frac{3\pi}{4} + n\pi,\ n \in I$

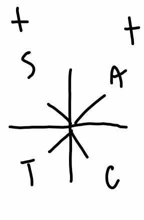
iii) $\csc^2 x - 3\csc x - 28 = 0,\ 0° \le x \le 360°$
Answer to the nearest degree

$(\csc x - 7)_{=0}(\csc x + 4)_{=0}$

$\csc x = 7$ $\csc x = -4$

$\sin x = \frac{1}{7}$ $\sin x = -\frac{1}{4}$

ref $\angle = 8°$ ref $\angle = 14°$



a) $x_1 = 8°$ $x_3 = 180 + 14° = 194°$

$x_2 = 180 - 8 = 172°$ $x_4 = 360 - 14 = 346°$

b) gen Soln $= 8° + 360°n,\ 172° + 360n,\ 194° + 360°n,\ 346° + 360n,\ n \in I$

iv) $2\cos^2\theta + 5\cos\theta - 3 = 0,\ 0 \le \theta \le \pi.$

$(2\cos^2\theta + 6\cos\theta)(-1\cos\theta - 3)$

$2\cos\theta(\cos\theta + 3) - 1(\cos\theta + 3)$

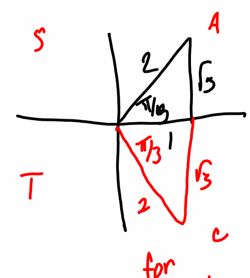
$(2\cos\theta - 1)_{=0}(\cos\theta + 3)_{=0}$

$\cos\theta = \frac{1}{2}$ $\cos\theta = -3$

ref $\angle = \frac{\pi}{3}$ No Soln

$\theta = \frac{\pi}{3}$

but $\theta_2 = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$



for general only

b) gen Soln $\frac{\pi}{3} + 2n\pi,\ \frac{5\pi}{3} + 2n\pi,\ n \in I$

Complete Assignment Questions #1 - #10

Assignment

1. Factor the following trigonometric expressions:

a) $4 \sin^2 \theta - \cos^2 \theta$ **b)** $\cot^2 x - \cot x$ **c)** $\cot^2\theta - 1$

d) $\sec x \sin^2 x - 0.25 \sec x$ **e)** $\sec^4\theta - 1$ **f)** $\sin^2 \theta + 3 \sin \theta + 2$

g) $4 \cos^2 A - 4 \cos A - 3$ **h)** $2 \sin^2 x - 7 \sin x + 6$

2. Consider the equation $2 \cos^2 x + 3 \cos x + 1 = 0$.

a) Use a **graphical** approach to find the solution to the equation where $0 \le x \le 2\pi$. Give solutions as exact values.

b) Use an **algebraic** approach to find the solution to the equation where $0 \le x \le 2\pi$. Give solutions as exact values.

c) State the general solution to the equation.

3. Consider the equation $2 \sin x \cos x = 3 \sin x$.

a) Use a **graphical** approach to find the solution to the equation where $0 \le x \le 2\pi$. Give solutions as exact values.

b) Use an **algebraic** approach to find the solution to the equation where $0 \le x \le 2\pi$. Give solutions as exact values.

c) State the general solution to the equation.

4. Algebraically find the solutions to the following trigonometric equations. Give solutions as exact values.

a) $2 \sin^2 \theta + \sin \theta = 0$ where $0 \le \theta \le 2\pi$

b) $2 \sin^2 x - \sin x = 1$ where $0 \le x \le 2\pi$

c) $\cot^2 A + \cot A = 0$ where $0 \le A \le \pi$

d) $2 \cos^2 x = \sqrt{3} \cos x$ where $0 \le x \le 2\pi$

5. Algebraically find the general solutions to the following trigonometric equations. Give solutions as exact values.

a) $2\csc^2\theta - 2 = 3\csc\theta$

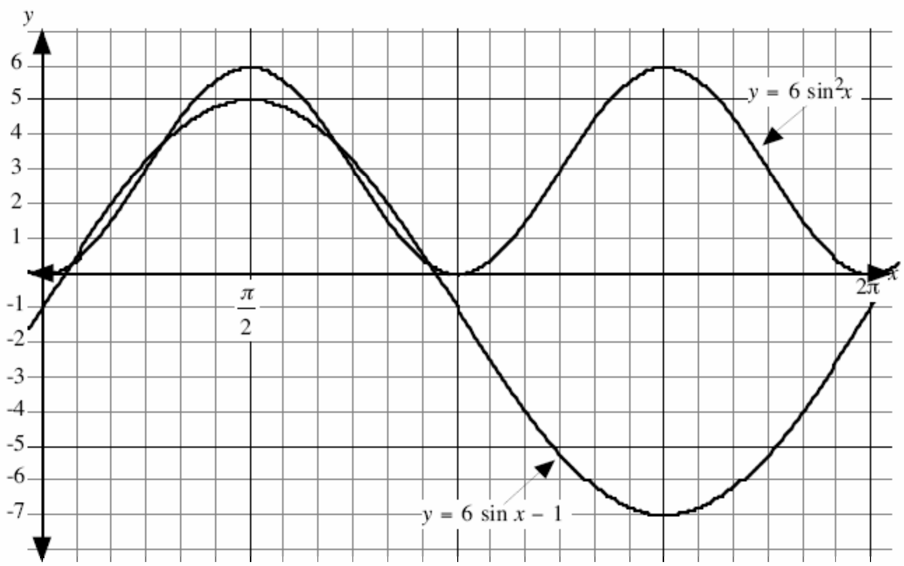
b) $3\sec\theta = 2 + \sec^2\theta$

6. Algebraically find the solutions to the following trigonometric equations, where $0 \leq x \leq 2\pi$. Give solutions in decimal form, correct the nearest hundredth.

a) $6\sin^2 x - 5\sin x = -1$

b) $2\sin x\cos x + 2\sin x = \cos x + 1$

7. The diagram below shows the graphs of the functions $y = 6\sin^2 x$ and $y = 6\sin x - 1$ where $0 \le x \le 2\pi$.



a) **Explain how** you could use this diagram to <u>estimate</u> the solution to the equation $6\sin^2 x - 6\sin x + 1 = 0$, where $0 \le x \le 2\pi$.

b) Use an algebraic approach to find the solutions to the equation $6\sin^2 x - 6\sin x + 1 = 0$, where $0 \le x \le 2\pi$.
Give the solution correct to the nearest hundredth.

c) **Explain how** you could use this diagram to <u>estimate</u> the solution to the equation $6\sin^2 x\,(6\sin x - 1) = 0$, where $0 \le x \le 2\pi$.

d) Use an algebraic approach to find the solutions to the equation $6\sin^2 x\,(6\sin x - 1) = 0$, where $0 \le x \le 2\pi$. Give the solution correct to the nearest hundredth.

Multiple Choice 8. Which solutions are correct for the equation $12\sin^2 x - 11\sin x + 2 = 0$?

A. $\sin x = 3, 8$

B. $\sin x = \frac{11}{12}, -2$

C. $\sin x = \frac{2}{3}, \frac{1}{4}$

D. $\sin x = -\frac{2}{3}, -\frac{1}{4}$

Numerical Response 9. The number of solutions of the equation $2\cos^2 x + \cos x - 1 = 0$, where $-8\pi \le x \le 8\pi$ is _____ .

10. If angle A is acute and $\log_4(\sin^2 A) = -1$, then the value of A, to the nearest tenth of a radian, is _____ .

Answer Key

1. **a)** $(2\sin\theta - \cos\theta)(2\sin\theta + \cos\theta)$ **b)** $\cot x(\cot x - 1)$ **c)** $(\cot\theta - 1)(\cot\theta + 1)$
d) $\sec x(\sin x + 0.5)(\sin x - 0.5)$ **e)** $(\sec\theta - 1)(\sec\theta + 1)(\sec^2\theta + 1)$
f) $(\sin\theta + 2)(\sin\theta + 1)$ **g)** $(2\cos A - 3)(2\cos A + 1)$ **h)** $(\sin x - 2)(2\sin x - 3)$

2. **a)** $\frac{2\pi}{3}, \pi, \frac{4\pi}{3}$ **b)** $\frac{2\pi}{3}, \pi, \frac{4\pi}{3}$ **c)** $x = \frac{2\pi}{3} + 2n\pi, \pi + 2n\pi, \frac{4\pi}{3} + 2n\pi, n \in I$

3. **a)** $0, \pi, 2\pi$ **b)** $0, \pi, 2\pi$ **c)** $x = n\pi, n \in I$

4. **a)** $0, \pi, \frac{7\pi}{6}, \frac{11\pi}{6}, 2\pi$ **b)** $\frac{\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$ **c)** $\frac{\pi}{2}, \frac{3\pi}{4}$ **d)** $\frac{\pi}{6}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{11\pi}{6}$

5. **a)** $x = \frac{\pi}{6} + 2n\pi, \frac{5\pi}{6} + 2n\pi, n \in I$ **b)** $x = 2n\pi, \frac{\pi}{3} + 2n\pi, \frac{5\pi}{3} + 2n\pi, n \in I$

6. **a)** 0.34, 0.52, 2.62, 2.80 **b)** 0.52, 2.62, 3.14

7. **a)** Find the x-coordinates of the points of intersection of the two graphs
b) 0.21, 0.91, 2.23, 2.93
c) Find the x-intercepts of each graph
d) 0.00, 0.17, 2.97, 3.14, 6.28

8. C **9.** 24 **10.** 0.5