Trigonometry -
Equations, Identities, and Modelling Lesson #4:
Solving Equations Involving Multiple Angles

Warm-Up #1

In this lesson we will be solving equations involving multiple angles, eg. $\sin 3x = 1$.

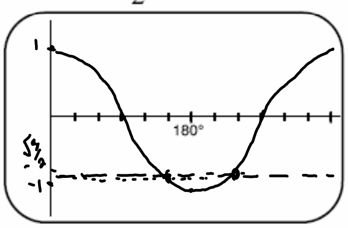
We will:

- introduce a graphical approach to determine an approximate solution to first degree trigonometric equations involving multiple angles
- find the **general solution** over the domain of real numbers
- algebraically find the solutions to an equation involving multiple angles.

Warm-Up #2 ***Graphically Exploring Solutions to Multiple Angle Equations***

Consider the equations $\cos x = -\frac{\sqrt{3}}{2}$ and $\cos 2x = -\frac{\sqrt{3}}{2}$ where x is in degree measure.

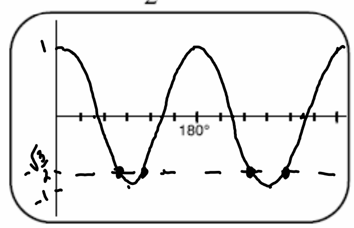
a) Use a graphical approach to determine the solution to the equation $\cos x = -\frac{\sqrt{3}}{2}$ where $0 \le x \le 360°$.



$x = 150°, 210°$

b) Use a graphical approach to determine the solution to the equation $\cos 2x = -\frac{\sqrt{3}}{2}$ where $0 \le x \le 360°$.

Period $= \frac{360°}{b}$ $= \frac{360°}{2}$ $= 180°$



$x = 75°, 255°$

$x = 105°, 285°$

c) Compare the solutions to the two equations in terms of:

- the number of solutions b) twice a)
- the values of x. 1st two values b) half 1st two values in a)

$\cos 2x$

$b = 2$ so factor $\frac{1}{2}$

d) Complete the following

i) The general solution to $\cos x = -\frac{\sqrt{3}}{2}$ is $150° + 360°n$, $210° + 360°n$, $n \in I$.

ii) The general solution to $\cos x = -\frac{\sqrt{3}}{2}$ consists of two sets of answers which differ by 360° degrees because the graph of $y = \cos x$ has a **period** of 360° degrees.

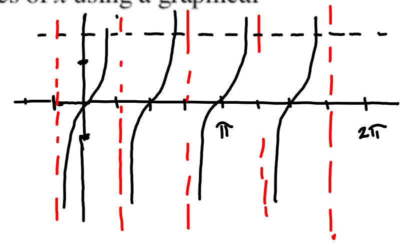
iii) The general solution to $\cos 2x = -\frac{\sqrt{3}}{2}$ will consist of two sets of answers which differ by 180° degrees because the graph of $y = \cos 2x$ has a **period** of 180° degrees.

iv) The general solution to $\cos 2x = -\frac{\sqrt{3}}{2}$ is $75° + 180°n$, $105° + 180°n$, $n \in I$.

Solving a Multiple Angle Equation Using a Graphical Approach

a) Given $\overset{y_1}{\tan 2x} = \overset{y_2}{\sqrt{3}}$, where $0 \le x \le 2\pi$, find the exact values of x using a graphical approach.

$x = \frac{\pi}{6}, \frac{2\pi}{3}, \frac{7\pi}{6}, \frac{5\pi}{3}$



b) State the general solution to the equation $\tan 2x = \sqrt{3}$.

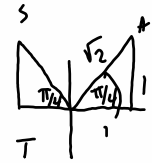
① $P = \frac{\pi}{b} = \frac{\pi}{2}$ ② $\frac{\pi}{6} + \frac{n\pi}{2}, n \in I$

c) Complete the following statement:

The general solution consists of answers which differ by $\underline{\pi/2}$ radians because the graph of $y = \tan 2x$ has a **period** of $\underline{\pi/2}$ radians.

Warm-Up #3 *Algebraically Exploring Solutions to Multiple Angle Equations*

$\sin x = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$



Consider the equation $\sin 3x = \frac{\sqrt{2}}{2}$.

a) Complete the following to solve the equation $\sin 3x = \frac{\sqrt{2}}{2}$, where $0 \le x \le 2\pi$

$\pi - \frac{\pi}{4} = \frac{3\pi}{4}$

- If x is defined for domain $0 \le x \le 2\pi$, then $3x$ is defined for domain $\underline{0} \le 3x \le \underline{6\pi}$

$\sin 3x = \frac{\sqrt{2}}{2}$ Quadrants $\underline{1}$ and $\underline{2}$

Reference angle =

$\overset{x}{3x} = \frac{\pi}{4}$ or $\frac{3\pi}{4}$ or $2\pi + \frac{\pi}{4}$ or $2\pi + \frac{3\pi}{4}$ or $4\pi + \frac{\pi}{4}$ or $4\pi + \frac{3\pi}{4}$

$\left(\frac{1}{3}\right)$ $3x = \frac{\pi}{4}$ or $\frac{3\pi}{4}$ or $\frac{9\pi}{4}$ or $\frac{11\pi}{4}$ or $\frac{17\pi}{4}$ or $\frac{19\pi}{4}$

$x = \frac{\pi}{12}$ or $\frac{\pi}{4}$ or $\frac{3\pi}{4}$ or $\frac{11\pi}{12}$ or $\frac{17\pi}{12}$ or $\frac{19\pi}{12}$

b) State the general solution to the equation $\sin 3x = \frac{\sqrt{2}}{2}$.

① $P = \frac{2\pi}{3}$ ② $\frac{\pi}{12} + \frac{2n\pi}{3}, \frac{\pi}{4} + \frac{2n\pi}{3}, n \in I$

c) Verify the solution using a graphical approach.

- The general solution consists of two sets of answers which differ by $\frac{2\pi}{3}$ radians because the graph of $y = \sin 3x$ has a **period** of $\frac{2\pi}{3}$ radians.

Solving a Multiple Angle Equation Using an Algebraic Approach

Use the following procedure to solve multiple angle equations

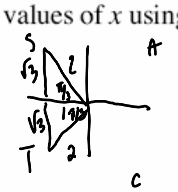
1. Find the domain for the multiple angle.
2. Solve for the multiple angle between 0 and 2π using the CAST rule and reference angle.
3. Add the period of the trigonometric graph of the multiple angle to each of the answers in step 2 until you cover the domain in step 1.

Consider the equation $\cos 2x = -\frac{1}{2}$.

$0 \le 2x \le 4\pi$

a) Find the exact values of x using an algebraic approach where $0 \le x \le 2\pi$. .

$\cos X = -\frac{1}{2}$



$X_1 = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$

$X_2 = \pi + \frac{\pi}{3} = \frac{4\pi}{3}$

$\left(\frac{1}{2}\right)2x = \frac{2\pi}{3} \times \left(\frac{1}{2}\right)$ $\quad$ $\left(\frac{1}{2}\right)2x = \frac{4\pi}{3} \times \left(\frac{1}{2}\right)$

$b = 2$

factor $\frac{1}{2}$

$x_1 = \frac{\pi}{3}$ $\quad$ $x_2 = \frac{2\pi}{3}$

$x_3 = \frac{\pi}{3} + \pi = \frac{4\pi}{3}$ $\quad$ $x_4 = \frac{2\pi}{3} + \pi = \frac{5\pi}{3}$

$P = \frac{2\pi}{b} = \frac{2\pi}{2} = \pi$

b) State the general solution to the equation $\cos 2x = -\frac{1}{2}$.

$\frac{\pi}{3} + n\pi, \quad \frac{2\pi}{3} + n\pi, \quad n \in I$

c) Complete the following statement.

The general solution consists of answers which differ by $\underline{\pi}$ radians because the graph of $y = \cos 2x$ has a **period** of $\underline{\pi}$ radians.

d) Verify the solution using a graphical approach.

Complete Assignment Questions #1 - #11

Assignment

1. a) Given $\sin 2x = \frac{\sqrt{3}}{2}$, where $0 \le x \le 2\pi$, find the exact values of x using a graphical approach.

b) Find the general solution to $\sin 2x = \frac{\sqrt{3}}{2}$.

2. a) Given $\cot 2x = 1$, where $0 \le x \le 2\pi$, find the exact values of x using a graphical approach.

b) Solve $\cot 2x = 1$ where $x \in \Re$.

3. Find the general solution to $\cos \frac{1}{2}x = \frac{\sqrt{3}}{2}$ using a graphical approach.

4. a) Use an algebraic approach to solve the equation $\sin 2x = \frac{\sqrt{2}}{2}$, $0 \le x \le 2\pi$

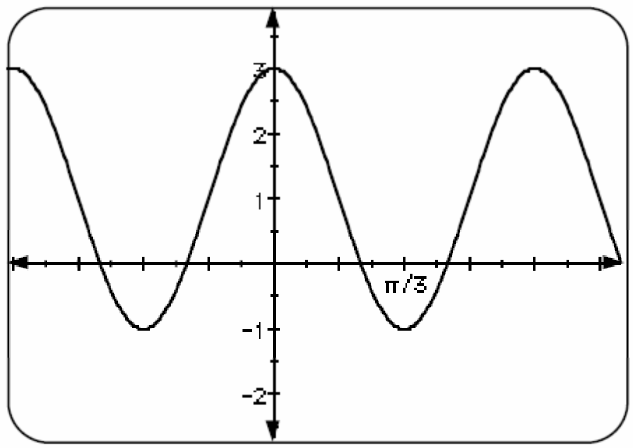
b) State the general solution to the equation $\sin 2x = \frac{\sqrt{2}}{2}$

5. a) Use an algebraic approach to solve the equation $\sec 3x = -2$, $0 \le x \le 2\pi$

b) State the general solution to the equation $\sec 3x = -2$

6. Use an algebraic approach to determine the general solution to the equation $\csc 2x = \dfrac{2\sqrt{3}}{3}$.

7. The graph of $y = 2\cos 3x + 1$, is displayed on a graphic calculator.



a) Describe the effects of the parameters 2, 3 and 1 on the graph of $y = \cos x$.

b) A student was asked to find all the values of θ which satisfy the equation $\cos 3x = -\dfrac{1}{2}, 0 \le x \le \pi$.

Explain how the student can find these values from the graph above and mark these points on the grid.

c) Show how to find these values by solving algebraically $\cos 3x = -\dfrac{1}{2}, 0 \le x \le \pi$.

8. a) Factor the expression $2\sin^2 x - \sin x - 1$ and hence solve the equation $2\sin^2 x - \sin x - 1 = 0$ for $0 \le x \le 2\pi$.

b) Describe how the solution of $2\sin^2\left(\frac{1}{2}x\right) - \sin\left(\frac{1}{2}x\right) - 1 = 0, 0 \le x \le 2\pi$ relates to the solution of $2\sin^2 x - \sin x - 1 = 0, 0 \le \theta \le 2\pi$. Find these solutions.

Multiple Choice

9. Which of the following is NOT a solution to the equation $2\sin 3x = 0$?

A. $\frac{\pi}{3}$

B. $\frac{\pi}{2}$

C. $\frac{4\pi}{3}$

D. 2π

10. If p and q are two solutions to the equation $\tan 5x = 0.8\pi$, which of the following statements CANNOT be true?

A. $p - q = 0.8\pi$

B. $p - q = \pi$

C. $p - q = 2.5\pi$

D. $p - q = 5\pi$

Numerical Response

11. The smallest positive solution to the equation $\sin 4x = 0.48$, correct to the nearest hundredth of a radian, is $x =$ _____ .

Answer Key

1.a) $\frac{\pi}{6}, \frac{\pi}{3}, \frac{7\pi}{6}, \frac{4\pi}{3}$ **b)** $x = \frac{\pi}{6} + n\pi, \frac{\pi}{3} + n\pi, n \in I$ **2.a)** $\frac{\pi}{8}, \frac{5\pi}{8}, \frac{9\pi}{8}, \frac{13\pi}{8}$ **b)** $x = \frac{\pi}{8} + \frac{n\pi}{2}, n \in I$

3. $x = \frac{\pi}{3} + 4n\pi, \frac{11\pi}{3} + 4n\pi, n \in I$ **4. a)** $\frac{\pi}{8}, \frac{3\pi}{8}, \frac{9\pi}{8}, \frac{11\pi}{8}$ **b)** $x = \frac{\pi}{8} + n\pi, \frac{3\pi}{8} + n\pi, n \in I$

5. a) $\frac{2\pi}{9}, \frac{4\pi}{9}, \frac{8\pi}{9}, \frac{10\pi}{9}, \frac{14\pi}{9}, \frac{16\pi}{9}$ **b)** $x = \frac{2\pi}{9} + \frac{2n\pi}{3}, \frac{4\pi}{9} + \frac{2n\pi}{3}, n \in I$

6. $x = \frac{\pi}{6} + n\pi, \frac{\pi}{3} + n\pi, n \in I$ **7. b)** Find x-intercepts of graph $y = 2\cos 3x + 1$ **c)** $\frac{2\pi}{9}, \frac{4\pi}{9}, \frac{8\pi}{9}$

8. a) $(2\sin x + 1)(\sin x - 1)$ and $x = \frac{\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$

b) the value of the solutions in a) will be doubled and any value outside the domain $0 \le x \le 2\pi$ will be disregarded. The solution is $x = \pi$.

9. B **10.** C **11.** 0.13