# Trigonometry - Equations, Identities, and Modelling Lesson #6: Trigonometry Identities Part 2

## Warm-Up

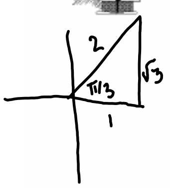
In this lesson we will verify and prove more complex trigonometric identities using the skills we learned from the previous lesson.
Some useful steps or hints when trying to prove a trigonometric identity are listed below.

## Hints in Proving an Identity

1. Begin with the more complex side.
2. If possible, use known identities given on the formula sheet, eg. try to use the Pythagorean identities when squares of trigonometric functions are involved.
3. If necessary change all trigonometric ratios to sines and/or cosines, eg. replace $\tan\theta$ by $\frac{\sin\theta}{\cos\theta}$, or $\sec\theta$ by $\frac{1}{\cos\theta}$.
4. Look for factoring as a step in trying to prove an identity.
5. If there is a sum or difference of fractions, write as a single fraction.
6. Occasionally, you may need to multiply the numerator or denominator of a fraction by its conjugate.


Consider the statement $\frac{1}{\cos\theta} - \cos\theta = \sin\theta\tan\theta$.

**a)** Verify the statement is true for $\theta = \frac{\pi}{3}$.




| L.S. | R.S. |
|---|---|
| $\frac{1}{\cos\frac{\pi}{3}} - \cos\frac{\pi}{3}$ = | $\sin\frac{\pi}{3}\tan\frac{\pi}{3}$ |
| $\frac{1}{\frac{1}{2}} - \frac{1}{2}$ = | $\left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{3}}{1}\right)$ |
| $2-\frac{1}{2}$ = | $\frac{\sqrt{9}}{2}$ |
| $\frac{3}{2}$ = | $\frac{3}{2}$ |

**b)** Use a graphing calculator to show that the statement is probably an identity.

$y_1 = \frac{1}{\cos\theta} - \cos\theta$ $\quad y_2 = \sin\theta\tan\theta$

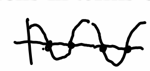
graphs are identical

**c)** Prove the statement is an identity using an algebraic approach

| L.S. | R.S. |
|---|---|
| $\frac{1}{\cos\theta} - \frac{\cos\theta\cdot\cos\theta}{1\cdot\cos\theta}$ = | $\sin\theta\cdot\frac{\sin\theta}{\cos\theta}$ |
| $\frac{1-\cos^2\theta}{\cos\theta}$ = | |
| $\frac{\sin^2\theta}{\cos\theta}$ = | $\frac{\sin^2\theta}{\cos\theta}$ |

**d)** State the restrictions in terms of $\theta$.

$\cos\theta \neq 0$



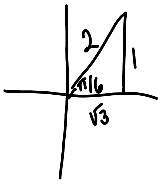
$\theta \neq \frac{\pi}{2} + n\pi,\ n\in I$

($\tan\theta \neq$ undefined, $\cos\theta \neq 0$ as above)

**Note that the restrictions can be verified from the graph.**


Consider the statement $\dfrac{\sin\theta}{\tan\theta} = \cos\theta$

**a)** Verify the statement is true for $\theta = \dfrac{\pi}{6}$




| L.S. | R.S. |
|---|---|
| $\dfrac{\sin\frac{\pi}{6}}{\tan\frac{\pi}{6}}$ | $\cos\frac{\pi}{6}$ |
| $\dfrac{\frac{1}{2}}{\frac{1}{\sqrt{3}}}$ | $\frac{\sqrt{3}}{2}$ |
| $\frac{1}{2}\cdot\frac{\sqrt{3}}{1} = \frac{\sqrt{3}}{2}$ | $\frac{\sqrt{3}}{2}$ |

**b)** Use a graphing calculator to show that the statement is probably an identity.

$y_1 = \dfrac{\sin\theta}{\tan\theta}$ $\quad y_2 = \cos\theta$

graphs are identical

**c)** Prove the statement is an identity using an algebraic approach

| L.S. | R.S. |
|---|---|
| $\dfrac{\sin\theta}{\frac{\sin\theta}{\cos\theta}}$ | $\cos\theta$ |
| $\sin\theta \times \dfrac{\cos\theta}{\sin\theta}$ | |
| $\cos\theta$ | $\cos\theta$ |

**d)** State the restrictions in terms of $\theta$.

$\tan\theta \neq 0$ $\quad \theta \neq n\pi, n \in I$

because $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ $\quad \tan\theta \neq$ undefined

$\theta \neq \frac{\pi}{2} + n\pi, n \in I$


Prove the following identity algebraically.

$\sin x\cos^2 x + \sin^3 x = \dfrac{1}{\csc x}$

| L.S. | R.S. |
|---|---|
| $\sin x(\cos^2 x + \sin^2 x)$ | $\sin x$ |
| $\sin x\ (1)$ | $\sin x$ |
| $\sin x$ | $\sin x$ |

**Class Ex. #4**

Prove that $\dfrac{\sin A + \tan A}{1 + \cos A} = \dfrac{1}{\cot A}$ is an identity.
For what values of $A$ is this identity undefined?

common denominator: $\dfrac{\dfrac{\sin A \cos A}{1 \cdot \cos A} + \dfrac{\sin A}{\cos A}}{1+\cos A}$ | $\dfrac{1}{\cot A}$

factor: $\dfrac{\dfrac{\sin A\cos A + \sin A}{\cos A}}{1+\cos A}$ | $\tan A$

mult by the reciprocal: $\dfrac{\sin A(\cos A + 1)}{\cos A} \times \dfrac{1}{1+\cos A}$ | $\tan A$

$\tan A$ | $\tan A$

Restrictions

$\cot A \neq 0$ $\tan A \neq$ undefined

$A \neq \frac{\pi}{2} + n\pi, n \in I$

$1+\cos A \neq 0$

$\cos A \neq -1$ $A \neq \pi + 2n\pi, n \in I$

$\tan A = \frac{\sin A}{\cos A}$ $\cos A \neq 0$ $A \neq \frac{\pi}{2} + n\pi, n \in I$

**Class Ex. #5**

Prove the identity $\dfrac{\sec^2\theta}{\sec^2\theta - 1} = \csc^2\theta$.

$\dfrac{\sec^2\theta}{\tan^2\theta} = \csc^2\theta$

$\dfrac{\frac{1}{\cos^2\theta}}{\frac{\sin^2\theta}{\cos^2\theta}} = \csc^2\theta$

$\frac{1}{\cos^2\theta} \times \frac{\cos^2\theta}{\sin^2\theta}$

$\frac{1}{\sin^2\theta} = \csc^2\theta$

$\csc^2\theta = \csc^2\theta$

LS = RS

## *Using Identities to Solve Equations*

**Class Ex. #6**

Solve the following equations where $0 \le x \le 2\pi$.

**a)** $2\cos^2 x - 3\sin x = 0$

$2(1-\sin^2 x) - 3\sin x = 0$

$2 - 2\sin^2 x - 3\sin x = 0$

$2\sin^2 x + 3\sin x - 2 = 0$

$-4 = 4 \times -1$

$3 = 4 + -1$

$(2\sin^2 x + 4\sin x)(-1\sin x - 2)$

$2\sin x(\sin x + 2) - 1(\sin x + 2)$

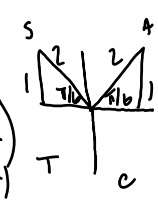
$\sin x + 2 = 0$ $2\sin x - 1 = 0$

$\sin x = -2$ $\sin x = \frac{1}{2}$

No Soln

$x = \frac{\pi}{6}$

$x = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$



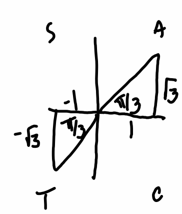

**b)** $\sin x - \sqrt{3}\cos x = 0$

$\frac{\sin x}{\cos x} = \frac{\sqrt{3}\cos x}{\cos x}$

$\tan x = \frac{\sqrt{3}}{1}$

$x = \frac{\pi}{3}$

$x = \pi + \frac{\pi}{3} = \frac{4\pi}{3}$




**Complete Questions #1 - #11**

# Assignment

**1.** Consider the statement $\frac{\cos\theta - \sin\theta}{\cos\theta} = \sin^2\theta - \tan\theta + \cos^2\theta$.

**a)** Verify the statement is true for $\theta = \frac{\pi}{4}$.

**b)** Prove the statement is an identity.

**c)** State, and give reasons, for any restrictions.

**2.** Consider the statement $\frac{\cot\theta - 1}{\tan\theta - 1} = -\frac{1}{\tan\theta}$.

**a)** Verify the statement is true for $\theta = \frac{\pi}{3}$

**b)** Prove the statement is an identity.

**c)** State, and give reasons, for any restrictions.

**3**. In each of the following :

**i)** verify the possibility of an identity using a graphing calculator
**ii)** prove the identity using an algebraic approach
**iii)** state any restrictions.

**a)** $\dfrac{\tan\theta\cos\theta}{\sin\theta} = 1$

**b)** $\sec^2 x - \sin^2 x = \cos^2 x + \tan^2 x$

**4**. Prove the following identities using an algebraic approach.

**a)** $(1 - \cos^2 x)(\csc x) = \sin x$

**b)** $(\sin\theta + \cos\theta)^2 = 1 + 2\sin\theta\cos\theta$

**c)** $\dfrac{1 - \cos x}{\sin x} = \dfrac{\tan x - \sin x}{\tan x \sin x}$.

**d)** $\dfrac{2}{1 - \sin\theta} + \dfrac{2}{1 + \sin\theta} = 4\sec^2\theta$

**e)** $\dfrac{1 + \cos x}{\tan x + \sin x} = \cot x$

**f)** $\sec x - \cos x = \dfrac{\sin x}{\cot x}$

**5.** Use conjugates to prove the following identities using an algebraic approach.

**a)** $\dfrac{1}{1 - \sin A} = \dfrac{1 + \sin A}{\cos^2 A}$

**b)** $\dfrac{1 - \cos A}{\sin A} = \dfrac{\sin A}{1 + \cos A}$

**6**. Prove that $\dfrac{2\cos x - 1}{2\cos^2 x - 7\cos x + 3} = \dfrac{1}{\cos x - 3}$ is an identity.

For what values of $x$ is this identity undefined?

**7.** Solve for $x$ as an exact value where $0 \leq x \leq 2\pi$.

**a**) $3 - 3\sin x - 2\cos^2 x = 0$ **b**) $\tan^2 x - 1 = \sec x$ **c**) $\sin^2 x - \cos^2 x = 0$

**8.** Solve, to the nearest tenth, the equations $7\sec^2 x + 2\tan x - 6 = 2\sec^2 x + 2, 0 \le x \le 2\pi$.

**Multiple Choice**

**9.** The identity $\dfrac{\sec x + 1}{\sec x - 1} + \dfrac{\cos x + 1}{\cos x - 1} = 0$ has restrictions

**A.** $x \ne 2n\pi, \dfrac{\pi}{2} + 2n\pi, n \in I$

**B.** $x \ne 2n\pi, \dfrac{\pi}{2} + n\pi, n \in I$

**C.** $x \ne \pi + 2n\pi, \dfrac{\pi}{2} + 2n\pi, n \in I$

**D.** $x \ne 2n\pi, n \in I$

**Numerical Response**

**10.** The value of $n$ to the nearest tenth for which the statement below is an identity, is _____ .

$$\frac{1 + \sin\theta}{\cos\theta} + \frac{\cos\theta}{1 + \sin\theta} = \frac{n}{\cos\theta}.$$

**11.** If $\dfrac{p}{2}\cos^2\dfrac{\pi}{5} + \dfrac{p}{2}\sin^2\dfrac{\pi}{5} = 4$, the value of $p$, to the nearest tenth, is _____ .

### *Answer Key*

**1.** **a)** both sides equal 0 **c)** $\theta \ne \dfrac{\pi}{2} + n\pi, n \in I$

**2.** **a)** both sides equal $-\dfrac{\sqrt{3}}{3}$ **c)** $\theta \ne n\pi, \dfrac{\pi}{4} + n\pi, \dfrac{\pi}{2} + n\pi, n \in I$ or $\theta \ne \dfrac{\pi}{4} + n\pi, n\dfrac{\pi}{2}, n \in I$

**3.** **a)** $\theta \ne n\pi, \dfrac{\pi}{2} + n\pi, n \in I$ or $\theta \ne n\dfrac{\pi}{2}, n \in I$ **b)** $x \ne \dfrac{\pi}{2} + n\pi, n \in I$

**6.** $x \ne \dfrac{\pi}{3} + 2n\pi, \dfrac{5\pi}{3} + 2n\pi, n \in I$ **7.a)** $\dfrac{\pi}{6}, \dfrac{\pi}{2}, \dfrac{5\pi}{6}$ **b)** $\dfrac{\pi}{3}, \pi, \dfrac{5\pi}{3}$ **c)** $\dfrac{\pi}{4}, \dfrac{3\pi}{4}, \dfrac{5\pi}{4}, \dfrac{7\pi}{4}$

**8.** 0.5, 2.4, 3.7, 5.5 **9.** B **10.** 2.0 **11.** 8.0