Trigonometry - Equations, Identities, and Modelling Lesson #7: Addition Identities

Warm-Up #1

Consider the statement $\sin(\alpha + \beta) = \sin \alpha + \sin \beta$.

a) Determine whether or not the statement can be verified using $\alpha = 60°$ and $\beta = 30°$.

$\sin(60° + 30°) = \sin 90 = 1$ $\quad \sin 60 + \sin 30 = \frac{\sqrt{3}}{2} + \frac{1}{2}$ $\quad LS \neq RS$

b) What can we say about the statement $\sin(\alpha + \beta) = \sin \alpha + \sin \beta$?

not true

Warm-Up #2

Use exact values to verify the following statements:

a) $\sin(60 + 30)° = \sin 60° \cos 30° + \cos 60° \sin 30°$

$\sin 90 = \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{1}{2} \cdot \frac{1}{2}$

$1 \qquad \frac{3}{4} + \frac{1}{4} = 1$

$LS = RS.$

b) $\sin\left(\frac{\pi}{3} - \frac{\pi}{6}\right) = \sin \frac{\pi}{3} \cos \frac{\pi}{6} - \cos \frac{\pi}{3} \sin \frac{\pi}{6}$

$\sin \frac{2\pi}{6} - \frac{\pi}{6} = \frac{\sqrt{3}}{2} \frac{\sqrt{3}}{2} - \frac{1}{2} \cdot \frac{1}{2}$

$\sin \frac{\pi}{6} \qquad \frac{3}{4} - \frac{1}{4} = \frac{2}{4} = \frac{1}{2}$

$1/2$

$LS = RS$ ✓

c) $\cos(30° + 60°) = \cos 30° \cos 60° - \sin 30° \sin 60°$

$\cos 90° = \frac{\sqrt{3}}{2} \cdot \frac{1}{2} - \frac{1}{2} \frac{\sqrt{3}}{2}$

$0 \qquad \frac{\sqrt{3}}{4} - \frac{\sqrt{3}}{4} = 0$

$LS = RS$ ✓

d) $\cos\left(\frac{\pi}{2} - \frac{\pi}{4}\right) = \cos \frac{\pi}{2} \cos \frac{\pi}{4} + \sin \frac{\pi}{2} \sin \frac{\pi}{4}$

$\cos\left(\frac{2\pi}{4} - \frac{\pi}{4}\right) = 0\left(\frac{1}{\sqrt{2}}\right) + 1\left(\frac{1}{\sqrt{2}}\right)$

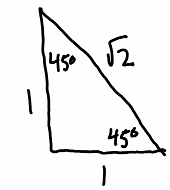
$LS = RS$

$\cos \frac{\pi}{4} \qquad 0 + \frac{1}{\sqrt{2}}$

$\frac{1}{\sqrt{2}} \qquad \frac{1}{\sqrt{2}}$

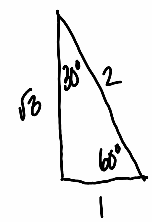
Addition Identities

Warm-Up #2 is an example of verifying the addition identities, sometimes called the sum and difference identities.



$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$
$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$
$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

These identities are on the formula sheet.



Use a difference identity to find the exact value of sin 15°. $= \sin(45-30)$

$\sin 45 \cos 30 - \cos 45 \sin 30$

$\frac{1}{\sqrt2}\cdot\frac{\sqrt3}{2} - \frac{1}{\sqrt2}\,\frac{1}{2}$

$\frac{\sqrt3}{2\sqrt2} - \frac{1}{2\sqrt2} = \frac{\sqrt3-1}{2\sqrt2}\,\frac{(\sqrt2)}{(\sqrt2)} = \frac{\sqrt6-\sqrt2}{4}$

Find the exact value of $\sec\frac{5\pi}{12}$. $\frac{1}{\cos\frac{5\pi}{12}}$ $\quad \cos\frac{5\pi}{12} = \cos\left(\frac{3\pi}{12}+\frac{2\pi}{12}\right) = \cos\left(\frac{\pi}{4}+\frac{\pi}{6}\right)$

$\cos\frac{\pi}{4}\cos\frac{\pi}{6} - \sin\frac{\pi}{4}\sin\frac{\pi}{6}$

$\frac{1}{\sqrt2}\,\frac{\sqrt3}{2} - \frac{1}{\sqrt2}\,\frac{1}{2}$

$\frac{\sqrt3}{2\sqrt2} - \frac{1}{2\sqrt2} = \frac{\sqrt3-1}{2\sqrt2} \xRightarrow{\sec} \frac{2\sqrt2}{\sqrt3-1}\,\frac{(\sqrt3+1)}{(\sqrt3+1)} = \frac{2\sqrt6+2\sqrt2}{3-1} = \frac{2\sqrt6+2\sqrt2}{2}$

$= \sqrt6+\sqrt2$

Simplify the following:

a) $\sin 100° \cos 10° - \cos 100° \sin 10°$ $\quad = \sin(100-10) = \sin 90 = 1$

b) $\cos\left(\frac{1}{4}\pi - \theta\right)\cos\left(\frac{1}{4}\pi + \theta\right) - \sin\left(\frac{1}{4}\pi - \theta\right)\sin\left(\frac{1}{4}\pi + \theta\right)$ $\quad = \cos\left[\left(\frac{\pi}{4}\cancel{-\theta}\right)+\left(\frac{\pi}{4}\cancel{+\theta}\right)\right]$

$\cos\frac{2\pi}{4} = \cos\frac{\pi}{2} = 0$

Class Ex. #4

Given $\cos\alpha = \frac{3}{5}$ and $\cos\beta = \frac{5}{13}$, where $0 \le \alpha \le \frac{\pi}{2}$ and $\frac{3\pi}{2} \le \beta \le 2\pi$, find the exact value of $\cos(\alpha + \beta)$.

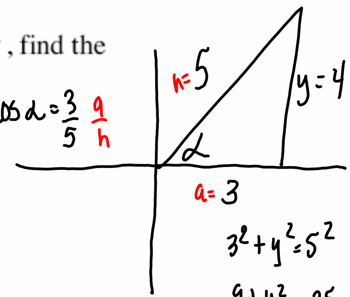
(Use sketch) $\left(\frac{a}{h}\right)$

$\cos\alpha\cos\beta - \sin\alpha\sin\beta$

$\frac{3}{5}\cdot\frac{5}{13} - \frac{4}{5}\left(\frac{-12}{13}\right)$

$\frac{15}{65} + \frac{48}{65} = \boxed{\frac{63}{65}}$

$\cos\alpha = \frac{3}{5}$ $\frac{a}{h}$



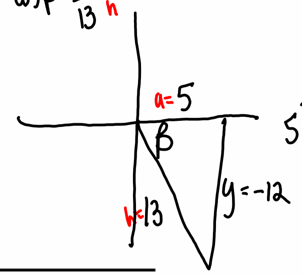
$3^2 + y^2 = 5^2$

$9 + y^2 = 25$

$y^2 = 16$

$y = 4$

$\cos\beta = \frac{5}{13}$ $\frac{a}{h}$



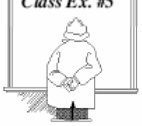
$5^2 + y^2 = 13^2$

$25 + y^2 = 169$

$y^2 = 144$

$y = 12$

Class Ex. #5



Consider the function $f(x) = \sin\left(\frac{\pi}{4} + x\right) - \sin\left(\frac{\pi}{4} - x\right)$.

a) Simplify $f(x)$.

$\cancel{\sin\frac{\pi}{4}\cos x} + \cos\frac{\pi}{4}\sin x - \left[\cancel{\sin\frac{\pi}{4}\cos x} - \cos\frac{\pi}{4}\sin x\right]$

$2\cos\frac{\pi}{4}\sin x$

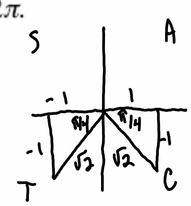
$2\cdot\frac{1}{\sqrt{2}}\sin x \frac{(\sqrt{2})}{(\sqrt{2})} = \frac{2\sqrt{2}\sin x}{2} = \sqrt{2}\sin x$

b) Use the result in a) to solve the equation $f(x) = -1$ where $0 \le x \le 2\pi$.

$\frac{\sqrt{2}\sin x}{\sqrt{2}} = \frac{-1}{\sqrt{2}}$

$\sin x = \frac{-1}{\sqrt{2}}$

ref $x = \frac{\pi}{4}$



$x = \pi + \frac{\pi}{4} = \frac{5\pi}{4}$ $\quad x = 2\pi - \frac{\pi}{4} = \frac{7\pi}{4}$

c) Verify the solutions in b) graphically.

$y_1 = \sqrt{2}\sin x$ $\quad y_2 = -1$ $\quad x$ coordinates of the pts of intersection

Complete Assignment Questions #1 - #12

Assignment

1. Simplify using the addition identities.

a) $\cos(180 - B)°$ **b)** $\sin\left(\frac{\pi}{2} - x\right)$

c) $\cos(90 + t)°$ **d)** $\sin(\pi + x)$

2. Simplify and evaluate the following:

a) $\sin 70° \cos 20° + \cos 70° \sin 20°$ **b)** $\cos 170° \cos 50° + \sin 170° \sin 50°$

c) $\sin \frac{\pi}{3} \cos \frac{\pi}{6} - \sin \frac{\pi}{6} \cos \frac{\pi}{3}$ **d)** $\sin^2\left(\frac{\pi}{2} - x\right) + \sin^2 x$

3. Use exact values to show that:

a) $\sin 75° = \frac{\sqrt{6} + \sqrt{2}}{4}$ **b)** $\cos \frac{\pi}{12} = \frac{\sqrt{6} + \sqrt{2}}{4}$

4. Find the exact value of each of the following:

a) $\cos \dfrac{13\pi}{12}$ **b)** $\csc 105°$

5. Given $\sin \alpha = \dfrac{3}{5}$ and $\sin \beta = \dfrac{7}{25}$, and α and β are both acute angles, show that $\cos(\alpha + \beta) = \dfrac{3}{5}$.

6. Given that $\tan X = \dfrac{12}{5}$ and $\tan Y = \dfrac{4}{3}$, where $0 \le X \le \dfrac{\pi}{2}$ and $\pi \le Y \le \dfrac{3\pi}{2}$, find the exact values for $\cos(X + Y)$ and $\sin(X + Y)$.

7. Consider the function $f(x) = \cos\left(x + \frac{\pi}{6}\right) - \cos\left(x - \frac{\pi}{6}\right)$.

a) Simplify $f(x)$.

b) Use the result in a) to solve the equation $f(x) = 1$ where $0 \leq x \leq 2\pi$.

c) Verify the solution(s) in b) graphically.

8. Prove the following identities:

a) $\dfrac{\sin(A - B)}{\cos A \cos B} = \tan A - \tan B$

b) $(\cos A + \cos B)^2 + (\sin A + \sin B)^2 = 2[1 + \cos(A - B)]$

c) $\sin(x + y)\sin(x - y) = \sin^2 x - \sin^2 y$

d) $\cos(x + y)\cos(x - y) = \cos^2 x + \cos^2 y - 1$

Multiple Choice **9.** If $\cos(\alpha + \beta) = 0.8320$ and $\cos(\alpha - \beta) = 0.4358$ then the value of $\cos\alpha\cos\beta$ is

A. 1.2678

B. 0.6339

C. 0.3962

D. 0.1981

10. The value of $\cos(\pi + y) - \cos(\pi - y)$ is

A. 0

B. 2

C. –2

D. Dependent on the value of y.

11. Given $\csc x = \frac{-17}{15}$ where $\frac{3\pi}{2} \leq x \leq 2\pi$ and $\cot y = -\frac{3}{4}$ where $\frac{\pi}{2} \leq y \leq \pi$, the value of $\cos(x - y)$ is

A. $-\frac{84}{85}$

B. $-\frac{36}{35}$

C. $\frac{84}{85}$

D. $\frac{36}{35}$

Numerical Response

12. If $\sin(A + B) = 0.75$ and $\sin(A - B) = 0.43$, then the value of $\sin B \cos A$, to the nearest hundredth, is _____ .

Answer Key

1. a) $-\cos B°$ **b)** $\cos x$ **c)** $-\sin t°$ **d)** $-\sin x$

2. a) 1 **b)** $-\frac{1}{2}$ **c)** $\frac{1}{2}$ **d)** 1

4. a) $\frac{-\sqrt{6} - \sqrt{2}}{4}$ **b)** $\frac{4}{\sqrt{6} + \sqrt{2}} = \sqrt{6} - \sqrt{2}$ **6.** $\frac{33}{65}$, $-\frac{56}{65}$

7. a) $-\sin x$ **b)** $\frac{3\pi}{2}$ **9.** B **10.** A **11.** A **12.** 0.16