Permutations and Combinations Lesson #2:
Factorial Notation and Permutations

Factorial Notation

Consider how many ways there are of arranging 6 different books side by side on a shelf.

In this example we have to calculate the product $6 \times 5 \times 4 \times 3 \times 2 \times 1$

In mathematics this product is denoted by 6! ("6 factorial" or "factorial 6")

In general $\quad n! = n(n-1)(n-2)(n-3) \ldots (3)(2)(1)$, where $n \in W$.

Warm-Up #1

a) Use the factorial key on a calculator to evaluate the following.

$6! =$ 720 $\qquad 9! =$ 362 880

b) To simplify a quantity like $\frac{10!}{7!}$ there are a variety of approaches:

i) By using the factorial key on your calculator

$$\frac{10!}{7!} = \frac{3\ 628\ 800}{5040} = 720 \quad \text{or} \quad \frac{10!}{7!} = 720$$

ii) By Cancellation

$$\frac{10!}{7!} = \frac{10 \times 9 \times 8 \times 7!}{7!} = \frac{10 \times 9 \times 8 \times \cancel{7} \times \cancel{6} \times \cancel{5} \times \cancel{4} \times \cancel{3} \times \cancel{2} \times \cancel{1}}{\cancel{7} \times \cancel{6} \times \cancel{5} \times \cancel{4} \times \cancel{3} \times \cancel{2} \times \cancel{1}} = 720$$

Class Ex. #1

Find the value of: **a)** $\frac{43!}{40!}$ $= \frac{43 \times 42 \times 41 \times \cancel{40!}}{\cancel{40!}} = 74046$ $\qquad$ **b)** $\frac{37!}{33!\,4!}$ $= \frac{37 \times 36 \times 35 \times 34 \times \cancel{33!}}{\cancel{33!}\, 4 \times 3 \times 2 \times 1}$

$= 66045$

Class Ex. #2

Simplify the following expressions

a) $\frac{n!}{(n-2)!}$ $= \frac{n(n-1)\cancel{(n-2)!}}{\cancel{(n-2)!}}$

$= n^2 - 1n$

b) $\frac{(n+3)!}{n!}$ $= \frac{(n+3)(n+2)(n+1)\cancel{(n)!}}{\cancel{n!}}$

$= (n+3)(n+2)(n+1)$

c) $\frac{n!}{n(n-1)}$ $\quad \frac{\cancel{n}\cancel{(n-1)}(n-2)!}{\cancel{n}\cancel{(n-1)}}$

Complete Assignment Questions #1 - #5

Warm-Up #2

An internet site's access code consists of three digits. Knowing the three digits is not enough to access the site. The digits have to be entered in the correct order. The ***order*** of the ***arrangement*** of the digits is important.

Joe cannot remember the access code, except that it contains the digits 3, 5 and 7.

- List all the arrangements of these three digits that Joe could use to determine the access code.

3,5,7   5,3,7   7,3,5

3,7,5   5,7,3   7,5,3

There are six possible arrangements to consider and only one of them will access the site. This type of arrangement, where the order is important, is called a **permutation**.

Permutations

An **arrangement** of a set of objects in which the **order of the objects <u>is</u> important** is called a **permutation**.

Permutations of "n" different objects taken all at a time

- List all the arrangements of the letters of the word **CAT**.
- Write the number of permutations in factorial notation.

C – A – T
C – T – A
A – C – T
A – T – C
T – C – A
T – A – C

$3 \times 2 \times 1 = 3!$

$= 6$

This is an example of the following general rule:

The number of permutations of "*n*" different objects taken all at a time is $n!$

How many permutations are there of the letters of the word:

a) REGINA   **b) KELOWNA**

a) $6! = 720$

b) $7! = 5040$

Permutations of "n" different objects taken "r" at a time (r ≤ n)

- Use the fundamental counting principle to determine how many three letter arrangements can be made from the letters of the word **GRAPHITE.**

$\underline{8}\ \underline{7}\ \underline{6} = 336$

In the example above we have found the number of permutations of 8 (n) objects taken 3 (r) at a time. This is denoted by ${}_8P_3$

$${}_8P_3 = 8 \times 7 \times 6 = \frac{8 \times 7 \times 6 \times 5!}{5!} = \frac{8!}{(8-3)!} = 336$$

This is an example of the following general rule:

The number of permutations of "*n*" different objects taken "r" at a time is

$$\boxed{{}_nP_r = \frac{n!}{(n-r)!}}\ *$$

This formula is on the formula sheet

Use the ${}_nP_r$ key on a calculator to evaluate ${}_8P_3$. Verify using factorials.

$$\frac{8!}{(8-3)!} \quad \frac{8!}{5!} = \frac{8 \times 7 \times 6 \times \cancel{5!}}{\cancel{5!}} = 336$$

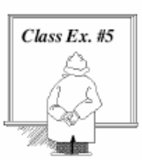
Defining 0!

If we replace r by n in the above formula we get the number of permutations of n objects taken n at a time. This we know is $n!$

$${}_nP_n = n! = \frac{n!}{(n-n)!} = \frac{n!}{0!}$$

For this to be equal to $n!$ the value of 0! must be 1.

0! is defined to have a value of 1.



In a South American country, vehicle license plates consist of any 2 different letters followed by 4 different digits. Find how many different license plates are possible by:

a) the fundamental counting principle

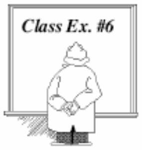
$$\underset{L}{\underline{26}}\ \underset{L}{\underline{25}}\ \underset{D}{\underline{10}}\ \underset{D}{\underline{9}}\ \underset{D}{\underline{8}}\ \underset{D}{\underline{7}} = 3\,276\,000$$

different license plates

b) permutations

$${}_{26}P_2 \cdot {}_{10}P_4 = (650)(5040) \quad 3276000$$

different license plates



Class Ex. #6

Solve for n in the equation $_nP_4 = 28 \times {}_{n-1}P_2$

$$\frac{n!}{(n-4)!} = 28 \times \frac{(n-1)!}{[(n-1)-2]!}$$

$(n-3)!$

$$\frac{n(n-1)(n-2)(n-3)\cancel{(n-4)!}}{\cancel{(n-4)!}} = \frac{28 \times (n-1)(n-2)\cancel{(n-3)!}}{\cancel{(n-3)!}}$$

$$n\cancel{(n-1)}\cancel{(n-2)}(n-3) = 28\cancel{(n-1)}\cancel{(n-2)}$$

$$n(n-3) = 28$$

$$n^2 - 3n = 28$$

$$n^2 - 3n - 28 = 0$$

$$\underbrace{(n-7)}_{=0}\underbrace{(n+4)}_{=0}$$

$n = 7$ accept $\cancel{n = -4}$ inadmissible

- In many cases involving simple permutations, the fundamental counting principle can be used in place of the permutation formulas.

Complete Assignment Questions #6 - #13

Assignment

1. Express as single factorials. (No work required.)

a) $6 \times 5 \times 4 \times 3 \times 2 \times 1$ **b)** $9 \times 8 \times 7 \times 6!$

c) $(n+2)(n+1)n(n-1) \ ...\times 3 \times 2 \times 1$

2. Find the value of the following:

a) $10!$ **b)** $\frac{8!}{4!}$ **c)** $\frac{15!}{10!\,5!}$ **d)** $\left(\frac{25!}{21!}\right)\left(\frac{7!}{11!}\right)$

3. Simplify each expression.

a) $\frac{n!}{n}$ **b)** $\frac{(n-3)!}{(n-2)!}$ **c)** $\frac{(n+1)!}{(n-1)!}$ **d)** $\frac{(3n)!}{(3n-2)!}$

4. Express as a quotient of factorials.

a) $9 \times 8 \times 7 \times 6$ **b)** $20 \times 19 \times 18$ **c)** $(n+2)(n+1)n$

5. Solve the equation.

a) $\dfrac{(n+1)!}{n\,!} = 6$ **b)** $(n+1)! = 6(n-1)!$

c) $\dfrac{(n+2)!}{n\,!} = 12$ **d)** $\dfrac{(n+1)!}{(n-2)!} = 20(n-1)$

6. How many arrangements are there of the letters:
a) DOG **b)** DUCK **c)** SANDWICH **d)** CANMORE ?

7. How many five-digit numbers can be made from the digits 2, 3, 4, 7 and 9 if no digit can be repeated?

8. If ${}_nP_r$ is the number of ways that n objects can be arranged r at a time, explain why ${}_7P_0 = 1$.

9. Use a permutation formula to determine how many arrangements there are of

a) two letters from the word **GOLDEN** **b)** three letters from the word **CHAPTERS**

c) four letters from the word **WEALTH**

10. How many numbers can be made from the digits 2, 3, 4, and 5 if no digit can be repeated? (Hint: consider 4 cases - four-digit numbers, three-digit numbers, two-digit numbers, one-digit numbers.)

11. Solve each equation, where n is an integer.

a) $\dfrac{n!}{84} = {}_{n-2}P_{n-4}$ **b)** ${}_{n}P_4 = 8({}_{n-1}P_3)$

Multiple Choice **12.** In a ten-team basketball league, each team plays every other team twice, once at home and once away. The number of games that are scheduled is

A. 45
B. 90
C. 100
D. 180

Numerical Response **13.** In a competition on the back of a cereal packet, seven desirable qualities for a kitchen (eg. spaciousness, versatility etc.) must be put in order of importance. The number of different entries that must be completed in order to ensure a winning order is _____ .

Answer Key

1. a) $6!$ b) $9!$ c) $(n+2)!$ **2.** a) 3 628 800 b) 1680 c) 3003 d) $\dfrac{115}{3}$

3. a) $(n-1)!$ b) $\dfrac{1}{n-2}$ c) $n(n+1)$ d) $3n(3n-1)$ **4.** a) $\dfrac{9!}{5!}$ b) $\dfrac{20!}{17!}$ c) $\dfrac{(n+2)!}{(n-1)!}$

5. a) $n = 5$ b) $n = 2$ c) $n = 2$ d) $n = 4$

6. a) 6 b) 24 c) 40 320 d) 5040

7. 120 **8.** ${}_7P_0 = \dfrac{7!}{(7-0)!} = \dfrac{7!}{7!} = 1$ **9.** a) 30 b) 336 c) 360

10. 64 **11.** a) $n = 7$ b) $n = 8$ **12.** B **13.** 5040