Permutations and Combinations Lesson #3:
Permutations with Restrictions; Permutations with Repetitions

Permutations with Restrictions

In many problems restrictions are placed on the order in which objects are arranged.
In this type of situation deal with the restrictions first.

In how many ways can all of the letters of the word **ORANGES** be arranged if:

a) there are no further restrictions?

$\underline{7}\ \underline{6}\ \underline{5}\ \underline{4}\ \underline{3}\ \underline{2}\ \underline{1} = 7!$ or ${}_7P_7$ $= 5040$

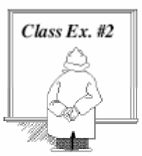
b) the first letter must be an **N**?

$\underset{N}{\underline{1}}\ \underline{6}\ \underline{5}\ \underline{4}\ \underline{3}\ \underline{2}\ \underline{1} = 1(6!)$ or $(1)\,{}_6P_6 = 720$

c) the vowels must be together in the order **O**, **A**, and **E**?

$\underset{OAE}{\underline{1}}\ \underline{4}\ \underline{3}\ \underline{2}\ \underline{1} \times 5$

$(1)(5)! = (1)(4!)(5) = 120$



In how many of the arrangements of the letters of the word **BRAINS** are the vowels together?

$\underset{AI}{\underline{1}}\ \underline{4}\ \underline{3}\ \underline{2}\ \underline{1}\ (5)$ or $\underset{IA}{\underline{1}}\ \underline{4}\ \underline{3}\ \underline{2}\ \underline{1}\ (5)$ $\quad 2!\,4!\,(5)$

$120 + 120 = 240$

Find the number of permutations of the letters in the word **KITCHEN** if:

a) the letters **K**, **C**, and **N** must be together but not necessarily in that order

$\underset{(KCN)}{(\underline{3}\ \underline{2}\ \underline{1})}\ \underline{4}\ \underline{3}\ \underline{2}\ \underline{1}\ (5)$

$3!\,(4)!\,5 =$

b) the vowels must not be together

No restrictions − Vowels are kept together

$7! \;-\; \underset{IE}{\underline{2!}}\ \underline{5}\ \underline{4}\ \underline{3}\ \underline{2}\ \underline{1}\ (6)$

$5040 - 1440 = 3600$

In how many ways can 3 girls and 4 boys be arranged in a row if no two people of the same gender can sit together?

$\underset{B}{\underline{4}}\ \underset{G}{\underline{3}}\ \underset{B}{\underline{3}}\ \underset{G}{\underline{2}}\ \underset{B}{\underline{2}}\ \underset{G}{\underline{1}}\ \underset{B}{\underline{1}} = (\text{Boys})(\text{Girls})$

$4!\ \ 3! = 144$

Six actors and eight actresses are available for a play with four male roles and three female roles. How many different cast lists are possible?

${}_6P_4 \cdot {}_8P_3 = (360)(336) = 120960$

How many odd five-digit numbers can be made with the following digits if no digits can be repeated?

a) 1, 2, 3, 4, 5, 6 and 7 (7)

6 5 4 3 4 = 1440
odd

b) 0, 2, 3, 4, 5, 6 and 7 (7)

5 5 4 3 3 = 900
≠0 0 is okay odd

Complete Assignment Questions #1 - #7

Permutations With Repetitions

In the previous section, the objects in each set were all different. But what happens when there are letters which repeat within the same word? To examine this scenario, consider the following four letter permutations of a word without repetitive letters, ROSE. Notice there are ${}_4P_4$, or 24 different arrangements.

ROSE	REOS	OSRE	SROE	SERO	EORS
ROES	RESO	OSER	SREO	SEOR	EOSR
RSOE	ORSE	OERS	SORE	EROS	ESRO
RSEO	ORES	OESR	SOER	ERSO	ESOR

Now, if we change the E in ROSE to a S, we get ROSS, a word with two letters which are repeating. If we change all the E's in the above list to S's, we will get all the arrangements for ROSS as shown in the list below.

ROSS	RSOS	OSRS	SROS	SSRO	SORS
ROSS	RSSO	OSSR	SRSO	SSOR	SOSR
RSOS	ORSS	OSRS	SORS	SROS	SSRO
RSSO	ORSS	OSSR	SOSR	SRSO	SSOR

Notice that the 24 original arrangements now become 12 different arrangements - 12 matching pairs of 2 four-letter arrangements. Arrangements like ROSE and ROES from the first list both become ROSS in the second list and count as only one arrangement.

There are $\frac{1}{2}$ or $\frac{1}{2!}$ as many permutations of ROSS as there are of ROSE.

Hence, the number of permutations of ROSS is $\frac{4!}{2!}$, or 12.

If we change the O and E in ROSE to S, we get RSSS, a "word" with three repeating letters, with the arrangements shown below.

RSSS	RSSS	SSRS	SRSS	SSRS	SSRS
RSSS	RSSS	SSSR	SRSS	SSSR	SSSR
RSSS	SRSS	SSRS	SSRS	SRSS	SSRS
RSSS	SRSS	SSSR	SSSR	SRSS	SSSR

Notice the 24 original arrangements now become 4 different arrangements - 4 matching sets of 6 four-letter arrangements. Arrangements like ROSE, ROES, RSOE, RSEO, REOS, and RESO from the first list all become one arrangement of RSSS.

There are $\frac{1}{6}$ or $\frac{1}{3!}$ as many permutations of RSSS as there are of ROSE.

Hence, the number of permutations of RSSS is $\frac{4!}{3!}$, or 4.

There is a pattern in the above lists If a letter appears twice in a word, we divide the total number of arrangements by 2!. If a letter appears three times in a word we divide the total number of arrangements by 3!.

The following formula gives the number of permutations when there are **repetitions.**

> The number of permutations of n objects, where a are the same of one type, b are the same of another type, and c are the same of yet another type, can be represented by the expression below
>
> $$\frac{n!}{a!\,b!\,c!}$$

Class Ex. #7

Find the number of permutations of the letters of the word:

a) **VANCOUVER** (9)

$\frac{9!}{2!} = 181440$

b) **MATHEMATICAL** (12)

$\frac{12!}{2!\,2!\,3!} = 19,958,400$

Class Ex. #8

How many arrangements of the word **POPPIES** can be made under each of the following conditions?

a) without restrictions

$\frac{7!}{3!} = 840$

b) if each arrangement begins with a **P**

$1\,(\frac{6!}{2!}) = 360$

c) if the first two letters are **P**

$(1)(5!) = 120$

d) if all the **P**'s are to be together

anywhere (5)

$(1)(4!) = 120$

e) if the first letter is **P** and the next one is not **P**

$1 \cdot 4 \cdot (\frac{5!}{2!}) = 240$

Class Ex. #9

Brett bought a carton containing 10 mini boxes of cereal. There are 3 boxes of Corn Flakes, 2 boxes of Rice Krispies, 1 box of Coco Pops, 1 box of Shreddies, and the remainder are Raisin Bran. Over a ten day period Brett plans to eat the contents of one box of cereal each morning.
How many different orders are possible if on the first morning he has Raisin Bran?

$1\,(\frac{9!}{3!\,2!\,2!}) = 15120$

Complete Assignment Questions #8 - #15

Assignment

1. How many arrangements could be made of the word:

a) **FATHER** if **F** is first?

b) **UNCLE** if **C** is first and **L** is last?

c) **DAUGHTER** if **UG** is last?

d) **MOTHER** if the vowels are first and last?

2. How many arrangements of the following words can be made if all the vowels must be kept together?

a) **FATHER**

b) **DAUGHTER**

c) **UNCLE**

d) **EQUATION**

3. Find the number of different arrangements of the letters in the word **ANSWER** under each condition:

a) without restrictions

b) that begin with an **S**

c) that begin with a vowel and end with a consonant

d) that have the three letters **A**, **N**, and **S** adjacent and in the order **ANS**

e) that have the three letters **A**, **N**, and **S** adjacent but not necessarily in that order

4. How many even four-digit numbers can be made from the digits 0, 2, 3, 4, 5, or 7 if no digit can be repeated?

5. Ann, Brian, Colin, Diane and Eric go to watch a movie and sit in 5 adjacent seats. In how many ways can this be done under each condition?

a) without restrictions?

b) if Brian sits next to Diane?

c) if Ann refuses to sit next to Eric?

6. In how many ways can four adults and five children be arranged in a single line under each condition?

a) without restrictions

b) if children and adults are alternated

c) if the adults are all together and the children are all together

d) if the adults are all together

7. Fifteen rugby players line up for a team picture, with seven players in the front row and eight players in the back row. How many different arrangements are possible under each conditions? Express the answer in factorial notation.

a) without restrictions

b) if the captain is in the middle of the front row

8. How many different arrangements can be made using all of the letters of each word?

a) **VICTORIA**

b) **ABBOTSFORD**

c) **LILLOOET**

d) **OSOYOOS**

9. Find the number of arrangements of the letters of the word **TATTOO** under each condition:

a) begins with a **T**

b) begins with two **T**'s

c) begins with three **T**'s

d) begins with one **T** and the next letter is not a **T**

e) begins with exactly two **T**'s

10. A town has 10 streets running from north to south and 8 avenues running from west to east. A man wishes to drive from the extreme south-west intersection to the extreme north-east intersection, moving either north or east along one of the streets or avenues. Find the number of routes he can take. (This question will be solved in a different manner in a lesson 7 of this unit).

11. How many permutations are there of the letters of the word **MONOTONOUS** under each condition?

a) without restrictions

b) if each arrangement begins with a **T**

c) if each arrangement begins with an **O**

d) if the four **O**'s are to be together

12. A race at the Olympics has 8 runners. In how many orders can their countries finish if there are:

a) 2 Canadian, 1 Russian, 1 German, 1 South African and 3 American athletes.

b) 1 Canadian, 2 British, 2 Ethiopian, 1 Algerian and 2 Kenyan runners

13. Naval signals are made by arranging coloured flags in a vertical line and the flags are then read from top to bottom. How many signals using six flags can be made if you have:

a) 3 red, 1 green and 2 blue flags? **b)** 2 red, 2 green and 2 blue flags?

c) unlimited supplies of red, green and blue flags?

Multiple Choice **14.** The number of different arrangements can be made using all the letters of the word **SASKATOON** is

A. 720

B. 45 360

C. 362 880

D. 725 760

Numerical Response **15.** The number of different ways that seven basketball players can be seated on a bench so that two specified players are always sitting side by side is _____ .

Answer Key

1. a) 120 b) 6 c) 720 d) 48

2. a) 240 b) 4320 c) 48 d) 2880

3. a) 720 b) 120 c) 192 d) 24 e) 144

4. 156 **5.** a) 120 b) 48 c) 72

6. a) 362 880 b) 2880 c) 5760 d) 17 280

7. a) 15! b) 14! **8.** a) 20 160 b) 907 200 c) 3360 d) 105

9. a) 30 b) 12 c) 3 d) 18 e) 9

10. $\frac{16!}{9! \cdot 7!} = 11\,440$ **11.** a) 75 600 b) 7560 c) 30 240 d) 2520

12. a) 3360 b) 5040 **13.** a) 60 b) 90 c) $3^6 = 729$ **14.** B **15.** 1440