# Permutations and Combinations Lesson 6:
## Pascal's Triangle and the Binomial Theorem

### Warm-Up #1

Study the following expansion of $(a+b)^n$, where $0 \le n \le 6$.

$$(a+b)^0 = 1$$
$$(a+b)^1 = a + b$$
$$(a+b)^2 = a^2 + 2ab + b^2$$
$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$
$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$
$$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$$
$$(a+b)^6 = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$$

You may want to do the expansions for yourself to verify the above results.

### Properties of the Expansion $(a+b)^n$

Use the above expansions to complete the following:

1. There are $n+1$ terms.
2. The sum of the exponents of $a$ and $b$ in each term is $n$.
3. The exponents of $a$ decreases term by term from $n$ to 0.
4. The exponents of $b$ increases term by term from 0 to $n$.
5. The coefficients in each expansion form a symmetrical arrangement.

### Warm-Up #2

The coefficients in the above expansion can be put in a triangular array known as Pascal's Triangle (named after Blaise Pascal who developed applications of the triangle in the seventeenth century.)

a) Use Warm-Up #1 to complete the 7th row of Pascal's Triangle.

b) Use the pattern in the triangle to complete the 8th and 9th row.

1 (1)
1 1 (2)
1 2 1 (3)
1 3 3 1 (4)
1 4 6 4 1 (5)
1 5 10 10 5 1 (6)
1 6 15 20 15 6 1 (7)
1 7 21 35 35 21 7 1 (8) *
1 8 28 56 70 56 28 8 1 (9)

c) Complete the following expansions:

$(a+b)^7 = 1a^7 + 7a^6b^1 + 21a^5b^2 + 35a^4b^3 + 35a^3b^4 + 21a^2b^5 + 7a^1b^6 + 1b^7$

$(a+b)^8 = 1a^2 + 8a^7b^1 + 28a^6b^2 + 56a^5b^3 + 70a^4b^4 + 56a^3b^5 + 28a^2b^6 + 8a^1b^7 + 1a^8$

## Warm-Up #3

Compare the combination array below with Pascal's Triangle.

$_0C_0$
$_1C_0 \quad _1C_1$
$_2C_0 \quad _2C_1 \quad _2C_2$
$_3C_0 \quad _3C_1 \quad _3C_2 \quad _3C_3$
$_4C_0 \quad _4C_1 \quad _4C_2 \quad _4C_3 \quad _4C_4$
$_5C_0 \quad _5C_1 \quad _5C_2 \quad _5C_3 \quad _5C_4 \quad _5C_5$
$_6C_0 \quad _6C_1 \quad _6C_2 \quad _6C_3 \quad _6C_4 \quad _6C_5 \quad _6C_6$
$_7C_0 \quad _7C_1 \quad _7C_2 \quad _7C_3 \quad _7C_4 \quad _7C_5 \quad _7C_6 \quad _7C_7$
$_8C_0 \quad _8C_1 \quad _8C_2 \quad _8C_3 \quad _8C_4 \quad _8C_5 \quad _8C_6 \quad _8C_7 \quad _8C_8$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1

a) Complete the next row in both tables.

b) Pascal derived the following theorem known as Pascal's Theorem.
Verify the theorem for $n = 5$ and $r = 1$.

$$\boxed{_{n+1}C_{r+1} = {}_nC_r + {}_nC_{r+1}}$$

$_{n+1}C_{r+1} = {}_nC_r + {}_nC_{r+1}$
$_{5+1}C_{1+1} = {}_5C_1 + {}_5C_{1+1}$
$_6C_1 = {}_5C_1 + {}_5C_2$
$15 = 5 + 10$
$15 = 15$
LS = RS ✓

## Warm-Up #4

***The Sum of the Numbers in the $k^{th}$ Row of Pascal's Triangle.***

$_0C_0$
$_1C_0 \quad _1C_1$
$_2C_0 \quad _2C_1 \quad _2C_2$
$_3C_0 \quad _3C_1 \quad _3C_2 \quad _3C_3$
$_4C_0 \quad _4C_1 \quad _4C_2 \quad _4C_3 \quad _4C_4$
$_5C_0 \quad _5C_1 \quad _5C_2 \quad _5C_3 \quad _5C_4 \quad _5C_5$
$_6C_0 \quad _6C_1 \quad _6C_2 \quad _6C_3 \quad _6C_4 \quad _6C_5 \quad _6C_6$

| | Row |
|---|---|
| 1 | 1 |
| 1 1 | 2 |
| 1 2 1 | 3 |
| 1 3 3 1 | 4 |
| 1 4 6 4 1 | 5 |
| 1 5 10 10 5 1 | 6 |
| 1 6 15 20 15 6 1 | 7 |

Complete the table which gives the sum of the numbers in each of the first six rows of Pascal's Triangle.

| Row | 1 | 2 | 3 | 4 | 5 | 6 | ... | $k$ |
|---|---|---|---|---|---|---|---|---|
| Sum | 1 | 2 | 4 | 8 | 16 | 32 | .... | |

a) Complete the following statement:

**"The sum of the numbers in the $k^{th}$ row of Pascal's Triangle is** $2^{k-1}$."

K = row

b) Look at row 7 in the diagram at the top of the page. This row has a sum of $2^6$. $= 64$

or $2^n$ = n exponent

Using the above rule we can see that

$_6C_0 + {}_6C_1 + {}_6C_2 + {}_6C_3 + {}_6C_4 + {}_6C_5 + {}_6C_6 = 2^6 = 64$

$1 + 6 + 15 + 20 + 15 + 6 + 1 = 64$

In general,

$$\boxed{_nC_0 + {}_nC_1 + {}_nC_2 + {}_nC_3 + \dots + {}_nC_{n-1} + {}_nC_n = 2^n}$$

## Warm-Up #5 — *The Sum of the Coefficients in the Expansion of $(a + b)^n$*

Consider the expansion of $(a+b)^n$.

$$(a+b)^1 = a + b$$

$$(a+b)^2 = a^2 + 2ab + b^2$$

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

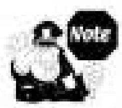
$$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$$

Complete the table which gives the sum of the coefficients in each of the first five expansions of $(a+b)^n$.

| $n$ | 1 | 2 | 3 | 4 | 5 | ... | $n$ |
|---|---|---|---|---|---|---|---|
| sum of coefficients | 2 | 4 | 8 | 16 | 32 | ... | $2^n$ |

Complete the following statement : $2^1$ $2^2$ $2^3$ $2^4$ $2^5$

**"The sum of the coefficients in the expansion of $(a + b)^n$ is $\boxed{2^n}$."**




**Memorize these results.**

- The sum of the numbers in the $k^{th}$ row of Pascal's Triangle is $\boxed{2^{k-1}}$
- ${}_nC_0 + {}_nC_1 + {}_nC_2 + {}_nC_3 + \dots + {}_nC_{n-1} + {}_nC_n = 2^n$
- The sum of the coefficients in the expansion of $(a+b)^n$ is $2^n$

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a) What is the sum of the numbers in the tenth $\underbrace{\text{row}}_{K}$ of Pascal's Triangle? $2^{K-1} = 2^{10-1} = 2^9 = 512$

b) What is the sum of the coefficients in the expansion of $(m+n)^{\boxed{12}}$? $n$ $2^n = 2^{12} = 4096$

c) What is the sum of: i) ${}_{15}C_0 + {}_{15}C_1 + {}_{15}C_2 + \dots + {}_{15}C_{15}$? K=16 or n=15 $2^n = 2^{15} = 32768$

ii) ${}_6C_0 + {}_6C_1 + {}_6C_2 + \dots + {}_6C_5$? n=6 or K=7 $2^n - {}_6C_6$

$2^6 - {}_6C_6 = 64 - 1 = 63$

**Complete Assignment Questions #1 - #6**

## Warm-Up #6 *Discovering the Binomial Theorem*

Referring to the combination array and Pascal's Triangle in Warm-Up #3, it is not just a coincidence that the combination symbols represent the numbers in Pascal's Triangle.

To investigate this consider the expansion of $(x + y)^6$.

$$(a+b)^6 = (a+b)(a+b)(a+b)(a+b)(a+b)(a+b)$$

$$= a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$$

Notice that each term in the expansion is a *combination* of $a$'s and $b$'s.

Consider how the third term (i.e.$15a^4b^2$) is formed. It is formed by choosing a $b$ from any two of the six factors in the expansion of $(a + b)^6$ and an $a$ from the remaining four factors. The two $b$'s can be chosen in ${}_6C_2$ (or 15) ways.
The four $a$'s can then be chosen in ${}_4C_4$ (or 1) way. This leads to the term ${}_6C_2a^4b^2$.

- Write the coefficients <u>in terms of combinations</u> for the expansion below.

$$(a+b)^6 = (a+b)(a+b)(a+b)(a+b)(a+b)(a+b)$$

$$= 1a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + 1b^6$$

$$= {}_6C_0a^b + {}_6C_1a^5b^1 + {}_6C_2a^4b^2 + {}_6C_3a^3b^3 + {}_6C_4a^2b^4 + {}_6C_5a^1b^5 + {}_6C_6b^6$$

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This leads to the following general formula for binomial expansions known as the Binomial Theorem.

## Binomial Theorem

$$(a+b)^n = {}_nC_0a^n + {}_nC_1a^{n-1}b + {}_nC_2a^{n-2}b^2 + \dots + {}_nC_ka^{n-k}b^k + \dots + {}_nC_nb^n, \text{ where } n \in I, n \geq 0$$

## General Term of the Expansion of $(a + b)^n$

The term ${}_nC_ka_{n-k}b^k$ is called the **<u>general</u> <u>term</u>** of the expansion.
It is the $(k+1)^{th}$ term in the expansion (not term $k$)

$$\boxed{t_{k+1} = {}_nC_k\, a^{n-k}\, b^k}$$

*This formula is on the formula sheet*

$(a+b)^n$

middle term

**Class Ex. #2**

Find: a) the fifth term of $(a+b)^8$  $n=8$

$t_5$

$t_{4+1}$

$k=4$

$t_5 = {}_nC_k\, a^{n-k} b^k$

$= {}_8C_4\, a^{8-4} b^4$

$= 70\, a^4 b^4$

b) the middle term of $(2a-5)^6$.

$t_4$

$t_{3+1}$

$k=3$

$n=6$

$t_4 = {}_nC_k\, a^{n-k} b^k$

$= {}_6C_3\, (2a)^{6-3} (-5)^3$

$= 20\, (8a^3)(-125)$

$= -20000\, a^3$

$\left({}_6C_0 \quad {}_6C_1 \quad {}_6C_2 \quad {}_6C_3 \quad {}_6C_4 \quad {}_6C_5 \quad {}_6C_6\right)$

$t_1 \quad t_2 \quad t_3 \quad t_4 \quad t_5 \quad t_6 \quad t_7$

$a = 2a$

$b = -5$

$n=6$

$\therefore$ 7 term

so $t_4$ is middle term

**Class Ex. #3**

Expand $(3x-2)^3$.  $a = 3x$  $b=-2$  $n=3$

${}_3C_0\,(3x)^3(-2)^0 + {}_3C_1\,(3x)^2(-2)^1 + {}_3C_2\,(3x)^1(-2)^2 + {}_3C_3\,(3x)^0(-2)^3$

$1(27x^3)(1) + (3)(9x^2)(-2) + (3)(3x)(4) + (1)(1)(-8)$

$27x^3 - 54x^2 + 36x - 8$

**Class Ex. #4**

One term in the expansion of $(x+a)^{10}$ is $3\,281\,250\,x^4$. Determine the numerical value of $a$.

$\boxed{a} = \boxed{x}$  $n=10$

$a^{n-k}$

$a^{10-k} = x^4$

$10-k=4$

$\therefore k=6$

${}_nC_k\, a^{n-k} b^k = 3\,281\,250\, x^4$

${}_{10}C_6\, x^{10-6} (a)^6 = 3\,281\,250\, x^4$

$\dfrac{210\, x^4\, a^6}{210x^4} = \dfrac{3\,281\,250\, x^4}{210x^4}$

$a^6 = 15\,625$

$\boxed{a=5}$

**Class Ex. #5**

Find the constant term (i.e. the term independent of $x$) in the expansion of $\left(2x - \dfrac{1}{x^2}\right)^{15}$.

$n=15$  16 terms

$k=k$

$-x^{-2}$

No variables so need $x^0$

${}_nC_k\, a^{n-k} b^k = {}_{15}C_k\, (2x)^{15-k} (-x^{-2})^k$

$= {}_{15}C_k\, 2^{15-k} \cdot x^{15-k} \cdot -1^k \cdot x^{-2k}$

$= {}_{15}C_k \cdot 2^{15-k} \cdot -1^k \cdot \left[x^{15-k} \cdot x^{-2k}\right]$ constant

$x^{15-3k} = x^0$

so $15-3k=0$

$-3k=-15$

$k=5$

Constant term

$= {}_{15}C_5\, (2x)^{15-5}(-1x^{-2})^5$

$= 3003\,(1024\, \cancel{x^{10}})(-1\, \cancel{x^{-10}})$

$= -3{,}075{,}072$

**Complete Assignment Questions #7 - #13**

# Assignment

**1.** Consider the expansion of $(a + b)^n$.

**a)** What are the coefficients of the first and last terms?

**b)** How many terms are there in the expansion?

**c)** What is the sum of the exponents in each term?

**d)** What is the coefficient of the third term?

**e)** The term in which position contains $b^4$?

**2.** What are the first three terms in the ninth row of Pascal's Triangle?

**3.** What are the last three terms in the sixteenth row of Pascal's Triangle?

**4.** Find the value of $n$ if the expansion of:

**a)** $(2x + 3)^n$ has 18 terms

**b)** $(3x - 5)^{4n-3}$ has 26 terms.

**5. a)** What is the sum of the terms in the ninth row of Pascal's Triangle?

**b)** What is the sum of the coefficients in the expansion of $(a + b)^9$?

**c)** What is the sum $_9C_0 + {_9C_1} + {_9C_2} + \ldots + {_9C_9}$?

**6.** Find the following sums :

**a)** $_{10}C_1 + {_{10}C_2} + {_{10}C_3} + \ldots + {_{10}C_{10}}$

**b)** $_7C_6 + {_7C_5} + {_7C_4} + \ldots + {_7C_1}$

**7.** Find the indicated term of each expansion.

**a)** the fifth term of $(a - b)^5$

**b)** the second term of $(x - 2)^6$

**c)** the third term of $(3x + 2y)^9$

**d)** the fourth term of $(a^2 - 2a)^7$

**e)** the middle term of $\left(2 - \frac{x}{2}\right)^6$

**8.** Expand and write in the simplest form:

**a)** $(2a + b)^3$

**b)** $\left(x - \frac{1}{x}\right)^4$

**9.** Find the indicated term of each expansion.

**a)** the term in $x^3$ in $(1 - 2x)^{12}$

**b)** the term in $x^4y^3$ in $(3x - y)^7$

**10. a)** One term in the expansion of $(x + a)^8$ is $448x^6$. Determine the value of $a$, $a > 0$.

**b)** One term in the expansion of $(x + b)^{11}$ is $-4455x^8$. Determine the value of $b$.

**11.** Find the indicated term of each expansion.

a) the constant term in the expansion of $\left(x^2 - \dfrac{1}{x}\right)^6$

b) the term independent of $x$ in $\left(2x + \dfrac{1}{x^3}\right)^8$

**Numerical Response**

**12.** If $(2a + 3)^{2n+3}$ has 18 terms, then the value of $n$ is _____ .

**13.** The term in $x^{11}$ in the expansion of $\left(x^2 + \dfrac{1}{x}\right)^{10}$ has a numerical coefficient, to the nearest whole number, of _____ .

### *Answer Key*

**1.** a) 1 b) $n + 1$ c) $n$ d) ${}_nC_2 = \dfrac{n(n-1)}{2}$ e) term 5

**2.** 1, 8, 28 **3.** 105, 15, 1 **4.** a) 17 b) 7

**5.** a) 256 b) 512 c) 512 **6.** a) 1023 b) 126

**7.** a) $5ab^4$ b) $-12x^5$ c) $314\,928\,x^7y^2$ d) $-280a^{11}$ e) $-20c^3$

**8.** a) $8a^3 + 12a^2b + 6ab^2 + b^3$ b) $x^4 - 4x^2 + 6 - \dfrac{4}{x^2} + \dfrac{1}{x^4}$ **9.** a) $-1760x^3$ b) $-2835x^4y^3$

**10.** a) 4 b) $-3$ **11.** a) 15 b) 1792 **12.** 7 **13.** 120