Probability Lesson #3:
Conditional Probability and the Event "A and B"

Warm-Up #1

One card is drawn from a deck of cards and is not replaced. A second card is then drawn.
Consider the following events
A = {the first card is a heart}
B = {the second card is a heart}

a) Determine $P(A)$ $= \frac{13}{52} = \frac{1}{4}$

b) Why is it not possible to determine $P(B)$?
The suit of the first card is not known

The probability of event B depends on whether or not event A occurred.
Events A and B are called **dependent events**.

Dependent Events

Two events are **dependent** if the knowledge that one event has occurred changes the probability of the other event occurring.

Warm-Up #2

One card is drawn from a deck of cards and is replaced. A second card is then drawn.
Consider the following events
A = {the first card is a heart}
B = {the second card is a heart}

a) Determine $P(A)$ $= \frac{13}{52} = \frac{1}{4}$

b) Determine $P(B)$ $= \frac{13}{52} = \frac{1}{4}$

The probability of event B does NOT depend on whether or not event A occurred.
Events A and B are called **independent events**.

Independent Events

Two events are **independent** if the knowledge that one event has occurred has no effect on the probability of the other event occurring.

Classify the following events as dependent or independent.

a) The experiment is rolling a die and tossing a coin.
The first event is rolling 2 on the die and the second event is tossing tails on the coin. Indep

b) The experiment is choosing two cards without replacement from a standard deck.
The first event is that the first card is a king and the second event is that the second card is a king. dep

c) The experiment is choosing two cards with replacement from a standard deck.
The first event is that the first card is a king and the second event is that the second card is a king. Indep

Conditional Probability

Consider the scenario from Warm-Up #1.
One card is drawn from a deck of cards and is not replaced. A second card is then drawn.
Consider the following events.

A = {the first card is a heart} $\qquad$ B = {the second card is a heart}

$P(A) = \dfrac{13}{52} = \dfrac{1}{4}$ but $P(B)$ depends on whether the event A occurred or did not occur.

- We denote by $P(B|A)$ the **conditional probability** of the event B, given that the event A has occurred.
  In this example $P(B|A) =$ $\frac{12}{51} = \frac{4}{17}$
- We denote by $P(B|\overline{A})$ the **conditional probability** of the event B, given that the event A has **NOT** occurred.
  In this example $P(B|\overline{A}) =$ $\frac{13}{51}$

Warm-Up #3

A red die and a blue die are tossed.
The outcomes are shown in the array.
Consider the following two events:

Event A {the sum of the two dice is 10}

Event B {the number on each die is the same}

Red Die \ *Blue Die*	1	2	3	4	5	6
1	{1,1}	{1,2}	{1,3}	{1,4}	{1,5}	{1,6}
2	{2,1}	{2,2}	{2,3}	{2,4}	{2,5}	{2,6}
3	{3,1}	{3,2}	{3,3}	{3,4}	{3,5}	{3,6}
4	{4,1}	{4,2}	{4,3}	{4,4}	{4,5}	{4,6}
5	{5,1}	{5,2}	{5,3}	{5,4}	{5,5}	{5,6}
6	{6,1}	{6,2}	{6,3}	{6,4}	{6,5}	{6,6}

6×6=36

a) List the outcomes for the following events as subsets of the sample space.

i) $A = \{(4,6), (5,5), (6,4)\}$

ii) $B = \{(1,1), (2,2), (3,3), (4,4), (5,5), (6,6)\}$

iii) A and $B = \{(5,5)\}$

b) State the following probabilities:

i) $P(A) = \frac{3}{36} = \frac{1}{12}$ **ii)** $P(B) = \frac{6}{36} = \frac{1}{6}$ **iii)** $P(B|A) = \frac{1}{3}$

iv) $P(A|B) = \frac{1}{6}$ **v)** $P(A \text{ and } B) = \frac{1}{36}$ **vi)** $P(B \text{ and } A) = \frac{1}{36}$

c) Verify the following:

i) $P(A \text{ and } B) = P(A) \times P(B|A)$

$\frac{1}{36} = \frac{1}{12} \times \frac{1}{3}$

$\frac{1}{36} = \frac{1}{36}$

ii) $P(A \text{ and } B) = P(B) \times P(A|B)$

$\frac{1}{36} = \frac{1}{6} \times \frac{1}{6}$

$\frac{1}{36} = \frac{1}{36}$

Multiplication Law for Dependent Events

Given that two events A, B, are dependent, then

$$P(A \text{ and } B) = P(A) \times P(B \mid A)$$

This formula is on the formula sheet

- The notation $P(B/A)$ is sometimes used instead of $P(B \mid A)$

Two cards are drawn **without** replacement from a standard deck of 52 cards. Determine the probability of the following events:

a) both cards are red

$P(RR) = \frac{26}{52} \times \frac{25}{51} = \frac{25}{102}$

b) neither card is a club

$P(\bar{C}\,\bar{C}) = \frac{39}{52} \times \frac{38}{51} = \frac{19}{34}$

c) the first card is a king and the second card is a five.

$P(K5) = \frac{4}{52} \times \frac{4}{51} = \frac{4}{663}$

d) one of the cards is a king and the other is a five.

$P(K5) \text{ or } P(5K) = \frac{4}{663} + \frac{4}{663} = \frac{8}{663}$

Multiplication Law for Independent Events

If the events A, B, are independent, then the knowledge that event A has occurred has no effect on the probability of the event B occurring.

This means that $P(B \mid A) = P(B)$.

Therefore we have the following law for independent events:

Given that two events A, B, are independent, then

$$P(A \text{ and } B) = P(A) \times P(B)$$

*This formula is **NOT** on the formula sheet*

Two cards are drawn **with** replacement from a standard deck of 52 cards. Determine the probability of the following events:

a) both cards are red

$P(RR) = \frac{26}{52} \times \frac{26}{52} = \frac{1}{4}$

b) the first card is a king and the second card is a five.

$P(K5) = \frac{4}{52} \times \frac{4}{52} = \frac{1}{169}$

c) one of the cards is a king and the other is a five.

$P(K5) \text{ or } P(5K) = \frac{1}{169} + \frac{1}{169} = \frac{2}{169}$

Often students confuse the concept of **mutually exclusive** events with that of **independent** events. These terms do NOT mean the same thing.

- The concept of **mutually exclusive events,** involves whether or not two events can occur simultaneously.
- The concept of **independent events** involves whether or not the occurrence of one event has an effect on the probability of the other event occurring.

If $P(A) = \frac{1}{3}$, $P(B) = \frac{2}{5}$ and $P(A \text{ or } B) = \frac{3}{5}$, investigate whether the events A and B are:

a) mutually exclusive events

$P(A \text{ or } B) = P(A) + P(B)$

$\frac{3}{5} = \frac{1}{3} + \frac{2}{5}$

$\frac{3}{5} = \frac{5}{15} + \frac{6}{15} = \frac{11}{15}$ ✗ Not M.E. so

b) independent events

$(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$

find "And" using "Or"

$= \frac{1}{3} + \frac{2}{5} - \frac{3}{5}$

$= \frac{5}{15} + \frac{6}{15} - \frac{9}{15} = \frac{2}{15}$

b) $P(A \text{ and } B) = P(A) \times P(B)$

$\frac{2}{15} = \frac{1}{3} \times \frac{2}{5}$

$\frac{2}{15} = \frac{2}{15}$ are independant

Let A and B be events with $P(A) = \frac{1}{2}$, $P(B) = \frac{1}{3}$, and $P(A \text{ and } B) = \frac{1}{4}$. Find:

a) $P(A|B)$

$P(A \text{ and } B) = P(B) \times P(A|B)$

$P(A|B) = \frac{P(A \text{ and } B)}{P(B)}$

$= \frac{1}{4} \div \frac{1}{3}$

$= \frac{1}{4} \times \frac{3}{1} = \frac{3}{4}$

b) $P(B|A) = \frac{P(A \text{ and } B)}{P(A)}$

$= \frac{1}{4} \div \frac{1}{2}$

$= \frac{1}{4} \times \frac{2}{1}$

$= \frac{2}{4} = \frac{1}{2}$

c) $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$

$= \frac{1}{2} + \frac{1}{3} - \frac{1}{4}$

$= \frac{6}{12} + \frac{4}{12} - \frac{3}{12} = \frac{7}{12}$

The probability that Ashley will pass Math this semester is 0.7 and the probability that she will pass English this semester is 0.9. If these events are independent, what is the probability (to the nearest hundredth) that she will pass

a) Math and English $= P(M) \times P(E)$

$= .7 \times .9 = .63$

b) Math or English $= P(M) + P(E) - P(M \text{ and } E)$

$= .7 + .9 - .63$

$= .97$

$M = .7$

$\overline{M} = .3$

$E = .9$

$\overline{E} = .1$

c) Math but not English

$P(M) \times P(\overline{E}) = .7 \times .1 = .07$

d) neither Math nor English

$P(\overline{M}) \times P(\overline{E}) = .3 \times .1 = .03$

Complete Assignment Questions #1 - #14

Assignment

1. Classify the following events as dependent or independent.

a) The experiment is to consider the height and weight of students.
The first event is that the student is greater than 1.8 m tall and the second event is that the student weighs more than 70 kg.

b) The experiment is choosing two cards with replacement from a standard deck.
The first event is that the first card is a jack and the second event is that the second card is a queen.

c) The experiment is choosing two cards without replacement from a standard deck.
The first event is that the first card is a seven and the second event is that the second card is a seven.

d) The experiment is rolling a die and rolling the die again.
The first event is that the number on the first roll is a six and the second event is that the number on the second roll is a two.

2. Consider two events such that $P(A) = \frac{1}{2}$, $P(B) = \frac{1}{3}$, and $P(A \text{ or } B) = \frac{3}{4}$.

a) are A, B mutually exclusive events? **b)** are A, B independent events?

3. Consider two events such that $P(A) = \frac{1}{4}$, $P(B) = \frac{2}{5}$, and $P(A \text{ and } B) = \frac{1}{10}$.

a) are A, B mutually exclusive events? **b)** are A, B independent events?

4. Let A and B be events with $P(A) = 0.6$, $P(B) = 0.4$, and $P(A \text{ and } B) = 0.3$. Find:
a) $P(A|B)$ **b)** $P(B|A)$ **c)** $P(A \text{ or } B)$

5. Let A and B be events with $P(A) = \frac{3}{8}$, $P(B) = \frac{5}{8}$, and $P(A \text{ or } B) = \frac{3}{4}$. Find:
a) $P(A|B)$ **b)** $P(B|A)$

6. A red die and a blue die are tossed. What is the probability that the red die shows a 1 and the blue die shows a 5 or a 6?

7. Two cards are drawn **with replacement** from a standard deck of 52 cards.
Determine the probability of the following events:
a) both cards are spades **b)** both cards are sevens **c)** neither card is red

d) the first card is a club and the second card is a diamond.

e) one of the cards is red and the other is black.

8. Two cards are drawn **without replacement** from a standard deck of 52 cards. Determine the probability of the following events:

a) both cards are spades **b)** both cards are sevens **c)** neither card is red

d) the first card is a club and the second card is a diamond.

e) one of the cards is red and the other is black.

9. The probabilities that Sara will pass grade 12 math and grade 12 physics this semester are 0.85 and 0.75 respectively. If these events are independent, what is the probability (to four decimal places) that she will pass:

a) both math and physics

b) math but not physics

c) physics but not math

d) neither math nor physics

10. In a city school, 60% of students have blue eyes, 55% have dark hair, and 20% have neither blue eyes nor dark hair.

a) Find the probability that a randomly selected student will have blue eyes and dark hair.

b) State with a reason if these two characteristics are independent.

Multiple Choice

11. Consider two events such that $P(A) = 0.3$, $P(B) = 0.4$, and $P(A \text{ or } B) = 0.58$. The events A, B are

A. Mutually exclusive and dependent

B. Not mutually exclusive and dependent

C. Mutually exclusive and independent

D. Not mutually exclusive and independent

12. A die is rolled and a coin is tossed. The probability of the die showing an even number and the coin coming up tails is

A. $\frac{1}{2}$

B. $\frac{1}{4}$

C. $\frac{1}{8}$

D. $\frac{1}{12}$

13. Four people roll a fair die to see who starts a game of scrabble. The probability that the first person throws a 2, the second person throws a 2, the third person throws a 4, and the fourth person throws a number other then 2 or 4 is

A. $\frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{4}{6}$

B. $\frac{1}{6} \times \frac{1}{6} \times \frac{1}{6} \times \frac{4}{6}$

C. $\frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{2}{6}$

D. $\frac{1}{6} \times \frac{1}{6} \times \frac{1}{6} \times \frac{2}{6}$

Numerical Response

14. A triangular spinner has 3 equal sections coloured red, blue, and green. The probability, to the nearest hundredth, of the spinner not landing on red on four consecutive spins is _____ .

Answer Key

1. **a)** dependent **b)** independent **c)** dependent **d)** independent

2. **a)** no **b)** no

3. **a)** no **b)** yes **4.** **a)** 0.75 **b)** 0.5 **c)** 0.7 **5.** **a)** $\frac{2}{5}$ **b)** $\frac{2}{3}$

6. $\frac{1}{18}$ **7.** **a)** $\frac{1}{16}$ **b)** $\frac{1}{169}$ **c)** $\frac{1}{4}$ **d)** $\frac{1}{16}$ **e)** $\frac{1}{2}$

8. **a)** $\frac{1}{17}$ **b)** $\frac{1}{221}$ **c)** $\frac{25}{102}$ **d)** $\frac{13}{204}$ **e)** $\frac{26}{51}$

9. **a)** 0.6375 **b)** 0.2125 **c)** 0.1125 **d)** 0.0375

10. **a)** 0.35 **b)** no, since $P(B \text{ and } D) \neq P(B)\,P(D)$ $\{0.35 \neq 0.33\}$

11. D **12.** B **13.** B **14.** 0.20