Probability Lesson #5:
Probability Problems Involving Conditional Probability

Warm-Up #1 *Review*

Recall the following probability formulas:

If the events A, B, are **mutually exclusive**, then $P(A \text{ or } B) = P(A) + P(B)$	If the events A, B, are **NOT mutually exclusive**, then $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$
If the events A, B, are **independent**, then $P(A \text{ and } B) = P(A) \times P(B)$	If the events A, B, are **dependent**, then $P(A \text{ and } B) = P(A) \times P(B \mid A)$

The formula for dependent events can be written as $P(A \mid B) = \dfrac{P(A \text{ and } B)}{P(A)}$

Warm-Up #2 *Review*

One card is drawn at random from a deck of 52 cards. The following events are defined:
A: a diamond is drawn $\frac{1}{4}$ B: a ten is drawn $\frac{1}{13}$ C: a red card is drawn $\frac{1}{2}$
Express the following probabilities as fractions in simplest form:

a) $P(\overline{A}) = \underline{3/4}$

$1 - P(A) = 1 - \frac{1}{4}$

b) $P(A \text{ or } C) = \underline{\frac{1}{2}}$ Not M.E.

$P(A) + P(C) - P(A \text{ and } C)$

$\frac{13}{52} + \frac{26}{52} - \frac{13}{52}$

c) $P(B \text{ and } C) = \underline{\frac{1}{26}}$

$P(B) \times P(C) = \frac{1}{13} \times \frac{1}{2} =$

d) $P(A \text{ and } \overline{B}) = \underline{3/13}$

$P(A) \times P(\overline{B}) = \frac{1}{4} \times \frac{\cancel{12}\,3}{13}$

e) $P(A \mid C) = \underline{1/4}$

$\dfrac{P(A \text{ and } C)}{P(C)} = \dfrac{\frac{1}{4} \times \frac{1}{2}}{\frac{1}{2}} = \frac{1}{8} \times \frac{2}{1} = \frac{1}{4}$

f) $P(C \mid A) = \underline{1/2}$

$\dfrac{P(A \text{ and } C)}{P(A)} = \dfrac{\frac{1}{8}}{\frac{1}{4}} = \frac{1}{8} \times \frac{4}{1} = 1/2$

Class Ex. #1

Two fair dice are rolled. Calculate the probability that 2 "ones" are rolled given that at least 1 "one" is rolled.

$P(\text{Two 1's} \mid \text{at least 1 one}) = \dfrac{P(\text{two 1's and at least one 1})}{P(\text{at least one 1})}$

$= \dfrac{P(\text{two 1s})}{P(\text{at least one 1})} = \dfrac{\frac{1}{6} \times \frac{1}{6}}{\frac{11}{36}} = \dfrac{\frac{1}{36}}{\frac{11}{36}} \boxed{= \frac{1}{11}}$

$P(\text{at least one 1})$

$1 - P(\text{no 1's})$

$1 - \left(\frac{5}{6} \times \frac{5}{6}\right)$

$= \frac{11}{36}$

Class Ex. #2

The table shows how the students in a large high school generally travel to school.

	Bus B	Car C	Other O	Total
Male, M	350	200	75	625
Female, F	300	175	100	575
Total	650	375	175	1200

a) Complete the totals in the chart.

b) How many students attended the high school?

1200

c) If a student is selected at random, determine the probability that

i) the student is female **ii)** the student travels by bus **iii)** the student is female and travels by bus

$P(F) = \frac{575}{1200} = \frac{23}{48}$ $P(B) = \frac{650}{1200} = \frac{13}{24}$ $P(F \text{ and } B) = \frac{300}{1200} = \frac{1}{4}$

d) Determine the probability that:

i) a female student travels by $\boxed{\text{bus.}}$ **ii)** a student who drives is $\boxed{\text{male}}$

$P(B|F) = \frac{300}{575} = \frac{12}{23}$ $P(C|m) = \frac{200}{375} = \frac{8}{15}$

e) Are the events "the student is female" and "the student travels by bus" independent events? Explain.

$P(F \text{ and } B)$	$P(F) \times P(B)$
$\frac{1}{4}$	$\frac{575}{1200} \times \frac{650}{120}$
$\frac{1}{4} \neq$	$\frac{23}{48} \times \frac{13}{24} = \frac{299}{1152}$

dependant

Class Ex. #3

As part of an experiment into the learning process, a mouse is put into a maze and rewarded with food every time it turns right. If a mouse turns right, the probability it turns right the next time is increased by 20%. If a mouse turns left, the probability it turns left the next time is decreased by 20%. Assuming that there is an equal probability that the first turn will be to the left or right, calculate the probability that:

a) the first two turns are both right

$P(R_1) = .5$ $P(R_2|R_1) = .5 + (.5)(.2) = .6$

$P(R_1 \text{ and } R_2) = (.5) \times (.6) = .3$

b) the first two turns are both left

$P(L) = .5$ $P(L_2|L_1) = .5 - (.5)(.2) = .4$

$P(L_1 \text{ and } L_2) = .5 \times .4 = \underline{\underline{.2}}$

c) the first two turns are different

$P(R_1 \text{ and } L_2)$ or $P(L_1 \text{ and } R_2)$

$(.5) \times (.4) + (.5) \times (.6)$

$.2 + .3 = .5$

d) the first three turns are all left

$P(L_1 \,\&\, L_2 \,\&\, L_3)$

$P(L_1)$ and $P(L_2|L_1)$ and $P(L_3|L_2 \text{ and } L_1)$

$(.5)(.4)[.4 - (.4)(.2)]$

$(.5)(.4)(.32) = 0.064$

Using a Probability Tree For Dependent Events

Cheryl is trying to show Jon how to solve problems based on the following information.

"Two machines A_1 and A_2 produce all the glass bottles made in a factory. Machine A_1 produces 60% of the output. The percentages of broken bottles produced by these machines are 5% and 8% respectively."

Cheryl suggests the following strategy.

First: Introduce symbols to represent the information.
Second: Write the given probabilities in terms of the symbols.
Third: Set up a tree diagram.

a) Complete Jon's work which is started below.

- A_1 - bottle is from machine A. A_2 - bottle is from machine A_2. B - bottle is broken.
- $P(A_1) = .6$ $P(A_2) = .4$ $P(B|A_1) = 0.05$ $P(B|A_2) = 0.08$

b) Jon sets up a probability tree diagram with the first branches leading to the machines and the second set of branches leading to the defective/non-defective items. Complete the diagram.

$P(A_1) = 0.6$ A_1

$P(B|A_1) = .05$ $B \rightarrow P(A_1 \text{ and } B) = P(A_1)\,P(B|A_1) = (.6)(.05) = .03$

$P(\overline{B}|A_1) = .95$ $\overline{B} \rightarrow P(A_1 \text{ and } \overline{B}) = P(A_1)\,P(\overline{B}|A_1) = (.6)(.95) = .57$

$P(A_2) = .4$ A_2

$P(B|A_2) = .08$ $B \rightarrow P(A_2 \text{ and } B) = P(A_2)\,P(B|A_2) = (.4)(.08) = .032$

$P(\overline{B}/A_2) = .92$ $\rightarrow P(A_2 \text{ and } \overline{B}) = P(A_2)\,P(\overline{B}|A_2) = (.4)(.92) = .368$

1.000

c) Cheryl asked Jon some questions to see if he understands what he has drawn.

i) Which of the final outcomes in the diagram relate to the event "a bottle is broken".

A_1 and B , A_2 and B

ii) If a bottle is chosen at random determine the probability that the bottle is broken.

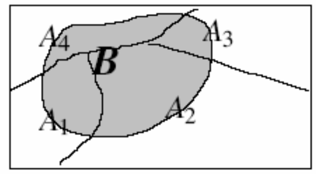
$P(B) = P(B|A_1) + P(B|A_2) = 0.03 + 0.032 = 0.062$

iii) Write a formula for $P(B)$ in terms of conditional probabilities.

$P(B) = P(A_1) \times P(B|A_1) + P(A_2) \times P(B|A_2)$

The formula on the previous page can be extended to include any number of machines A_1, A_2, A_3, etc.

In general if a sample space is partitioned into mutually exclusive outcomes $A_1, A_2, A_3 \ldots$ and if B is any other event then



$P(B) = P(A_1 \text{ and } B) + P(A_2 \text{ and } B) + P(A_3 \text{ and } B) + \ \ldots$

hence

$$\boxed{P(B) = P(A_1)\,P(B\,|\,A_1) + P(A_2)\,P(B\,|\,A_2) + P(A_3)\,P(B\,|\,A_3) + \ \ldots}$$

This formula is called the formula on total probability and is NOT given on the formula sheet.

Consider the following problem.

" Bag A contains 5 yellow and 5 green marbles. Bag B contains 7 yellow and 3 green marbles. One of the bags is chosen by selecting one card at random from a deck of cards. If a heart is selected, then a marble is taken at random from Bag A. If a heart is not selected, then a marble is taken from Bag B.

What is the probability that the marble is yellow?"

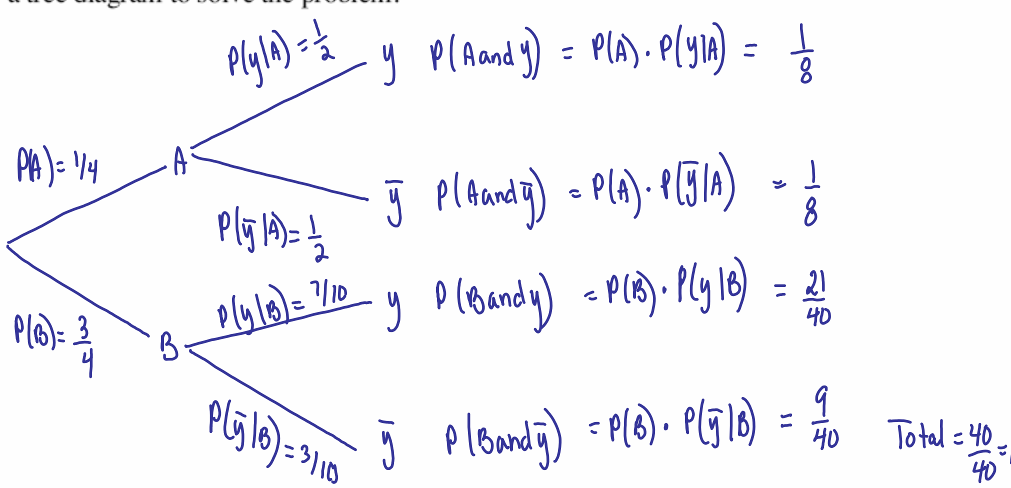
a) Complete:

$P(A) =$ 1/4 $\quad P(B) =$ 3/4 $\quad P(Y\,|\,A) =$ 5/10 = 1/2 $\quad P(Y\,|\,B) =$ $\frac{7}{10}$

b) Rewrite the formula on total probability using the symbols in a) and solve the problem.

$$P(y) = P(A \text{ and } y) + P(B \text{ and } y)$$
$$= P(A)\,P(y\,|A) + P(B)\,P(y\,|B)$$
$$= \frac{1}{4}\cdot\frac{1}{2} + \frac{3}{4}\cdot\frac{7}{10} = \frac{26}{40} = \frac{13}{20}$$

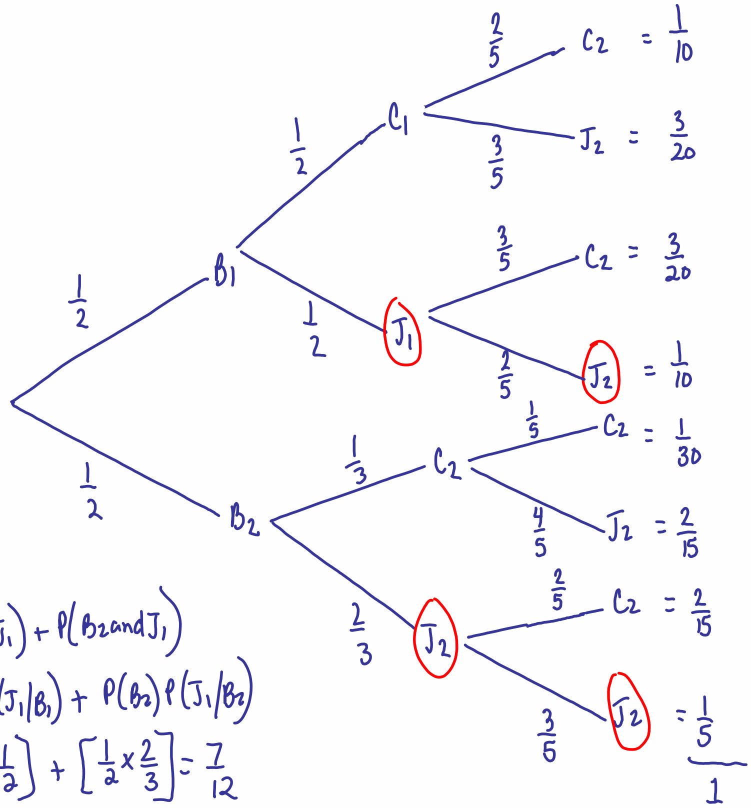
c) Use a tree diagram to solve the problem.



$$P(y) = P(A \text{ and } y) + P(B \text{ and } y) = \frac{1}{8} + \frac{21}{40} = \frac{5}{40} + \frac{21}{40} = \frac{26}{40} = \frac{13}{20}$$

Two boxes each contain 6 doughnuts. The first box contains 3 chocolate doughnuts and 3 jelly doughnuts. The second box contains 2 chocolate doughnuts and 4 jelly doughnuts. One of the boxes is selected at random and a doughnut is removed. It is a jelly doughnut. A second doughnut is then removed from the same box.

Calculate the probability that it is a jelly doughnut.



$$P(J_1) = P(B_1 \text{ and } J_1) + P(B_2 \text{ and } J_1)$$
$$= P(B_1)\cdot P(J_1/B_1) + P(B_2)P(J_1/B_2)$$
$$\left[\frac{1}{2}\times\frac{1}{2}\right] + \left[\frac{1}{2}\times\frac{2}{3}\right] = \frac{7}{12}$$

$P(J_2|J_1)$ = the probability that a jelly doughnut is removed second given a jelly doughnut has already been removed.

$$P(J_2|J_1) = \frac{P(J_2 \text{ and } J_1)}{P(J_1)} = \frac{\frac{1}{10}+\frac{1}{5}}{\left[\frac{1}{2}\times\frac{1}{2}\right]+\left[\frac{1}{2}\times\frac{2}{3}\right]}$$ see tree diagram $$= \frac{\frac{3}{10}}{\frac{7}{12}} = \frac{3}{10}\times\frac{12}{7} = \frac{18}{35}$$

Complete Assignment Questions #1 - #9

Assignment

1. Two cards are drawn with replacement from a deck of cards. Calculate the probability that two queens are drawn given that at least one queen is drawn.

2. An octagonal die numbered 1-8 is rolled twice. What is the probability of rolling only one 8, given that at least one 8 is rolled?

3. The table shows the distribution of blood types for students in the first year at a local college.

   a) Complete the totals in the chart.

   b) How many students are in first year at the college?

	O	A	B	AB	Total
Male, M	210	174	74	42	
Female, F	315	261	111	63	
Total					

   c) If a student is selected at random, determine the probability, to four decimal places, that the student;

   i) is male   ii) has blood type A   iii) is male and has blood type A

   d) Are the events "the student is male" and "the student has blood type A" independent events? Explain.

   e) Determine the probability, to four decimal places, that:

   i) a female student has blood type B   ii) a female student does not have blood type O

   iii) a student with blood type A is male   iv) a student with blood type AB is female.

4. Packet 1 contains 8 red balloons and 4 blue balloons. Packet 2 contains 5 red balloons, 3 green balloons and 4 blue balloons. A die is rolled. If the result is a one or a six then a balloon is chosen at random from packet 1. Otherwise, a balloon is chosen from packet 2. Calculate the probability that the chosen balloon is

a) blue **b)** green

5. In a diagnostic test for a disease, sometimes a positive reaction is obtained, even when the disease is not present. In the past the test has given a positive reaction to 90% of people having the disease and to 5% of people who do not have the disease. If 10% of the population have the disease, calculate the probability that a person chosen at random would not react positively to the test.

6. Three machines A, B and C produce respectively 50%, 30% and 20% of the items produced daily by a manufacturing company. The percentages of defective items produced by the machines are respectively 5%, 2% and 1%. What is the probability that an item selected at random from the daily output is defective?

7. A golfer is practicing putting. The probability that he holes the first putt is 0.4. Each time he holes a putt the probability that he holes the next putt increases by 25%. Each time he misses a putt the probability that he misses the next putt increases by 20%. Calculate the probability that:

a) he holes the first two putts

b) he misses the first two putts

c) he holes two of the first three putts

8. A box contains 10 cans of cola and 6 cans of lemonade. A second box contains 8 cans of cola and 8 cans of lemonade. One of the boxes is chosen at random and a can is selected at random from that box. The selected can is a can of lemonade. A second can is then selected from the same box.

Determine the probability that the second can is also lemonade.

9. A packet of candy contains 10 individually wrapped pieces of candy, each of which is either orange flavoured or lemon flavoured. One particular packet contains 8 orange and 2 lemon candies and a second packet contains 7 orange and 3 lemon candies. One of these packets is chosen at random and a candy is selected at random from that packet. It is lemon flavoured. A second piece of candy is then selected from the same packet. Calculate the probability that the second piece of candy is orange flavoured.

Answer Key

1. $\frac{1}{25}$ **2.** $\frac{14}{15}$

3. **b)** 1250 **c)** **i)** 0.4000 **ii)** 0.3480 **iii)** 0.1392
d) independent since $P(M \text{ and } A) = P(M) \times P(A)$
e) **i)** 0.1480 **ii)** 0.5800 **iii)** 0.4000 **iv)** 0.6000

4. **a)** $\frac{1}{3}$ **b)** $\frac{1}{6}$ **5.** 0.865 **6.** 0.033

7. **a)** 0.2 **b)** 0.432 **c)** 0.2138

8. $\frac{43}{105}$ **9.** $\frac{37}{45}$